Conversation 24: Properties of the Linear Span

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MATH 3200: Applied Linear Algebra

Question 43.1

Bob: It says here in Question 1 of Module 43:
"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n, \vec{\mathbf{w}}$ be any vectors such that $\vec{\mathbf{w}}$ is in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$. Prove that then for any scalar λ, the vector $\lambda\vec{\mathbf{w}}$ is also in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$."
Shall we give this one a try?

Denny: Oh no! A proof again ...
I have long forgotten what that *span* thing was.
What does it mean in plain English?

Bob: Let's look up the relevant definition of Lecture 22.

The linear span

Definition

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ be a set of vectors of the same order.
The *linear span* of these vectors is the set $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ of all linear combinations of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$.

Denny: Can we also look up "linear combination"?

Bob: If we don't remember, then we should. Here it is:

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \ldots, d_n$, called *coefficients* here, such that
$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

The first two steps

Bob: Now recall the question:

"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n, \vec{\mathbf{w}}$ be any vectors such that $\vec{\mathbf{w}}$ is in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$. Prove that then for any scalar λ, the vector $\lambda\vec{\mathbf{w}}$ is also in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$."

Cindy: I think we should begin the proof by writing:

"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n, \vec{\mathbf{w}}$ be any vectors such that $\vec{\mathbf{w}}$ is in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$."

Alice: Good start, Cindy!

Cindy: Should I then write next: "Then $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$. Let $d_1, d_2, \ldots, d_n$ be coefficients such that $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$."

Theo: Perfect!

Cindy: Now what do I do? I never know how to do these proofs!

Question C24.1: What should Cindy do next?

Translating the conclusion

Alice: We talked about this in Conversation 6. You have translated the assumptions, now you need to translate the conclusion of the theorem.

Cindy: Yes, now I remember! So, I should write next:
"We want to show that $\lambda\vec{\mathbf{w}}$ is also in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$."

Frank: What's that λ?

Cindy: I'm so sorry! I should have written:
"Let λ be any scalar. We want to show that $\lambda\vec{\mathbf{w}}$ is also in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$." Right?

Theo: Exactly! Now translate the statement you want to prove.

Cindy: "We want to show that $\lambda\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$."

Question C24.2: Did Cindy get this right?

Alice: Not quite. This would mean that the coefficients of the linear combination for $\lambda\vec{\mathbf{w}}$ are exactly the same as for $\vec{\mathbf{w}}$.
Is that what you want to prove?

Translating the conclusion, continued

Bob: You should have written:

"We want to show that there exist coefficients $d_1, d_2, \ldots, d_n$ such that $\lambda \vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$."

Cindy: Thank you, Bob! But shouldn't I rather have written:

"We want to show that there exist coefficients $c_1, c_2, \ldots, c_n$ such that $\lambda \vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n$."

With different letters? So as not to get mixed up when these coefficients are different?

Question C24.3: Whose notation is better, Bob's or Cindy's?

Alice: Excellent choice Cindy! Never use the same letter for two numbers, vectors, or matrices that might be different.

Cindy: Should I now do the calculations?

Denny: What calculations?

Calculating with the symbols

Cindy: Like this:

$\lambda\vec{\mathbf{w}} = \lambda(d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n) = \lambda d_1\vec{\mathbf{v}}_1 + \lambda d_2\vec{\mathbf{v}}_2 + \cdots + \lambda d_n\vec{\mathbf{v}}_n$."

Can we now make $c_1 = \lambda d_1, c_2 = \lambda d_2, \ldots, c_n = \lambda d_n$?

Frank: Why not?

Cindy: And what should I do next?

Denny: Nothing more. Put $\square$ to mark the end of your proof.

Cindy: Really? Is this the end of the proof?

Theo: Basically, yes. Formally, you would write:

"By choosing $c_1 = \lambda d_1, c_2 = \lambda d_2, \ldots, c_n = \lambda d_n$ we have found coefficients $c_1, c_2, \ldots, c_n$ such that $\lambda\vec{\mathbf{w}}$ is the linear combination $\lambda\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n$. This shows that $\lambda\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$, which means that $\lambda\vec{\mathbf{w}}$ is in the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$. $\square$ "

Cindy's complete proof at a glance

Denny: I got a little lost. Can we look at the entire proof?

Cindy: Of course! The students in this course will neatly write it up for you as the first part of their group work exercise.

Question 43.2

Bob: It says here in Question 2 of Module 43:

"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n, \vec{\mathbf{u}}, \vec{\mathbf{w}}$ be any vectors such that $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ are in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$. Prove that then the vector $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is also in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$."

Denny: Looks the same to me as the previous one.

Bob: So give it a try then!

Denny: Who, me? No way! Let Cindy do it.

Cindy: You helped me so nicely with the last step of the previous one, Denny! I'm sure you can do this one.

Theo: Don't be such a chicken Denny!

Denny: (grudgingly) Let's start the proof by writing:

"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n, \vec{\mathbf{u}}, \vec{\mathbf{w}}$ be any vectors such that $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ are in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$."

The next two steps

Alice: Good start, Denny!

Denny: Next we should write: "Then $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ are linear combinations of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$. Let $d_1, d_2, \ldots, d_n$ be coefficients such that $\vec{\mathbf{u}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$ and $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$."

Question C24.4: Did Denny get it right so far?

Cindy: You followed the right pattern, but you cannot assume that the coefficients for $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are the same, so you need to use different letters.

Denny: OK, forgot about that. So let's write:

"... Let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be coefficients such that $\vec{\mathbf{u}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n$ and $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$. We want to show that $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is also in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$."

Translating the conclusion, continued

Denny: Next we translate this into linear combinations:

"We want to show that there exist coefficients such that $\vec{\mathbf{u}} + \vec{\mathbf{w}} = \dots$."

Which letters should I use now, Cindy?
$c_1, c_2, \dots, c_n$ or $d_1, d_2, \dots, d_n$?

Question C24.5: What advice will Cindy give here?

Cindy: You should use new letters. Like $b_1, b_2, \dots, b_n$.

Denny: Thank you, Cindy! So let's write:

"We want to show that there exist coefficients $b_1, b_2, \dots, b_n$ such that $\vec{\mathbf{u}} + \vec{\mathbf{w}} = b_1\vec{\mathbf{v}}_1 + b_2\vec{\mathbf{v}}_2 + \dots + b_n\vec{\mathbf{v}}_n$."

Now what?

Question C24.6: What should Denny do next?

Cindy: Calculate $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ in the notation that you have set up.

Calculating with the symbols

Denny: (sigh) OK:

"$\vec{\mathbf{u}} + \vec{\mathbf{w}} = (c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n) + (d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n) =$"

Now what?

Cindy: You can factor out the vectors $\vec{\mathbf{v}}_i$ and rewrite this as:

"$\vec{\mathbf{u}} + \vec{\mathbf{w}} = (c_1 + d_1)\vec{\mathbf{v}}_1 + (c_2 + d_2)\vec{\mathbf{v}}_2 + \cdots + (c_n + d_n)\vec{\mathbf{v}}_n$."

Denny: Thank you, Cindy! Now let's write as Theo had told us:

"By choosing $b_1 = c_1 + d_1, b_2 = c_2 + d_2, \dots, b_n = c_n + d_n$ we have found coefficients $b_1, b_2, \dots, b_n$ such that $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is the linear combination $\vec{\mathbf{u}} + \vec{\mathbf{w}} = b_1\vec{\mathbf{v}}_1 + b_2\vec{\mathbf{v}}_2 + \cdots + b_n\vec{\mathbf{v}}_n$. This shows that $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$, which means that $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is in the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$. $\square$"

Theo: Congratulations, Denny! You did it!

Denny: (pleased) Well—with a little help from Cindy.

Denny's complete proof at a glance

Bob: Congratulations, Cindy and Denny!

Cindy: Thank you, Bob!

Denny: And the students in this course will now neatly write it up for us as the second part of their group work exercise.

Question 43.3

Bob: It says here in Question 3 of Module 43:

"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ be any vectors of the same order. Prove that any linear combination of two vectors in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ is also in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$."

Cindy: We could start again with

"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ be any vectors of the same order. Let $\vec{\mathbf{w}}$ be any linear combination of two vectors in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$."

But how do we write this in symbols?

Denny: Looks like a toughy to me.

Bob: Hmm. Maybe Theo could help us?

Theo: Will be happy to. You need to assume that

$\vec{\mathbf{w}} = a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2$,

where a, b are some scalars, and $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$ are in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$.

Cindy: So let's write:

"Let $\vec{\mathbf{w}} = a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2$, where $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$ are in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$."

Now we can express $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$ as linear combinations of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ as before: "Let $\vec{\mathbf{u}}_1 = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n$ and $\vec{\mathbf{u}}_2 = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$."

Should I calculate $a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2$ by substituting these expressions for $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ and then factoring out $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$? I mean:

$$a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2 = a(c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n) + b(d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n)$$

$$a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2 = (ac_1 + bd_1)\vec{\mathbf{v}}_1 + (ac_2 + bd_2)\vec{\mathbf{v}}_2 + \cdots + (ac_n + bd_n)\vec{\mathbf{v}}_n.$$

Question C24.7: Does Cindy's approach work?

Alice: This works and give us coefficients for the linear combination as before.

Frank: Forget it! There is nothing to prove in Question 43.3.

Bob: Don't be so negative Frank!!

A proof? What proof?

Frank: On the contrary! I'm very positive: We don't need a proof.

Denny: Sounds good to me. But why not?

Frank: Look: If $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$ are in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$, then $a\vec{\mathbf{u}}_1, b\vec{\mathbf{u}}_2$ are also in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$, as we have already shown in the proof for Question 43.1. But then the sum $\vec{\mathbf{w}} = a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2$ must also be in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$, as we have already shown in the proof for Question 43.2. So we don't need a proof that $a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2$ is in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$.

Alice: We do need a proof, and we have just seen two of them.

Bob: What are you talking about, Alice? Cindy gave us a very nice proof, but where have we seen another proof?

Question C24.8: What was Alice referring to?

Alice: Frank's argument. That's another proof.

This works for 3 vectors as well

Theo: Frank deduced from the facts that *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ *is closed under multiplication by scalars* (that's what we proved Question 43.1) and that *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ *is closed under addition of vectors* (that's what we proved in Question 43.2) that any linear combination of 2 vectors $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$ in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ is also in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$.

Denny: Wow!

Frank: If you say so. But the same reasoning also works for 3 vectors: Assume $\vec{\mathbf{w}} = a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2 + c\vec{\mathbf{u}}_3$, where $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$ are in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$. Then $a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2$ and $c\vec{\mathbf{u}}_3$ are in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ by what we have just shown and by the proof for Question 43.1. So $\vec{\mathbf{w}} = (a\vec{\mathbf{u}}_1 + b\vec{\mathbf{u}}_2) + c\vec{\mathbf{u}}_3$ must also be in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$, because the linear span is closed under vector addition, to use Theo's phrase.

And so on, for any number of vectors in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$.

Question 43.4

Cindy: Are we allowed to write "and so on" in a proof?

Theo: Not in a formal proof. But one can often first find a not entirely formal argument that uses "and so on," and then rewrite it as a formal proof using a technique that is called *proof by mathematical induction.* This formal translation is quite straightforward for Frank's argument. I can show you how—

Alice: Maybe not now. This proof technique goes a little beyond the scope of this course. I think Frank's argument is clear enough so that we can accept it here as a solution for Question 43.4

Frank: The one that said "challenge?" No way!

All: Congratulations, Frank!

Take-home message

Consider the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ of some set of vectors. Here it has been proved that

- $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ *is closed under multiplication by scalars*, which means that if $\vec{\mathbf{w}}$ is in this set and λ is any scalar, then $\lambda\vec{\mathbf{w}}$ is also in this set,
- $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ *is closed under addition of vectors*, which means that if $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ are in this set, then $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is also in this set,
- $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ *is closed under linear combinations*, which means that if $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_m$ are in this set, then any linear combination $d_1\vec{\mathbf{u}}_1 + d_2\vec{\mathbf{u}} + \cdots + d_m\vec{\mathbf{u}}_m$ is also in this set.

An issue that came up repeatedly in the proofs is that one needs to make sure to use different symbols for entities such as numbers, vecors, or matrices that might be different.