# Conversation 25A: Introduction to Linear Dependence and Linear Independence, Part I

Winfried Just
Department of Mathematics, Ohio University

MATH 3200: Applied Linear Algebra

# Question 43.6

**Bob:** It says here in Question 6 of Module 43:

"Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ be any vectors in $\mathbb{R}^3$. Is then the linear span *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ always a plane in $\mathbb{R}^3$?"

Did any of you find the answer?

**Question C25.1:** What was your answer?

**Cindy:** I think it must be "yes." When we take the span of 1 vector in $\mathbb{R}^3$, we get a line, when we take the span of 2 vectors in $\mathbb{R}^3$, we get a plane, when we take the span of 3 vectors in $\mathbb{R}^3$, we get the whole space $\mathbb{R}^3$. This is such a neat pattern:

When we take the span of $k$ vectors, we get a space of dimension $k$.

**Denny:** Nice pattern! Yeah, but when we take the span of 4 vectors in $\mathbb{R}^3$, we would get a 4-dimensional subspace.

How would that look?

# A counterexample

**Frank:** Wouldn't look like anything. There are no 4-dimensional subspaces of the 3-dimensional Euclidean space $\mathbb{R}^3$.
Cindy's pattern must be wrong!

**Alice:** The pattern that Cindy noticed actually works most of the time, but not always.

**Cindy:** Does it always work for dimensions 1, 2, and 3?

**Bob:** I think Question 6 of Module 43 is asking precisely whether it does work for dimension 2.

**Frank:** And the answer is "no."

**Bob:** Why?

**Frank:** Consider $\vec{\mathbf{v}}_1 = [1, 0, 0]$ and $\vec{\mathbf{v}}_2 = [-2, 0, 0]$.

**Question C25.2:** What is *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ in Frank's example?

# What if $\vec{\mathbf{v}}_2$ is a scalar multiple of $\vec{\mathbf{v}}_1$?

**Frank:** When $\vec{\mathbf{v}}_1 = [1, 0, 0]$ and $\vec{\mathbf{v}}_2 = [-2, 0, 0]$, then *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ comprises all vectors on the $x$-axis, but nothing more.
This set is a line, not a plane.

**Denny:** But $\vec{\mathbf{v}}_2 = -2[1, 0, 0] = -2\vec{\mathbf{v}}_1$.
So, the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ in your example are practically the same.
Your example is cheating, man!

**Frank:** It is a fine example of two different vectors.
What do you mean by "practically the same" anyway?

**Alice:** Good question, Frank!

**Cindy:** I think Denny means that when $\vec{\mathbf{v}}_2 = c\vec{\mathbf{v}}_1$ for some scalar $c$, then *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ can only be a line. Is that right?

**Bob:** Let's see. If $\vec{\mathbf{w}}$ is in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$, then $\vec{\mathbf{w}}$ is a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ so that for some scalars $d_1, d_2$:

$$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 = d_1\vec{\mathbf{v}}_1 + d_2(c\vec{\mathbf{v}}_1) = (d_1 + cd_2)\vec{\mathbf{v}}_1.$$

So every vector $\vec{\mathbf{w}}$ in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ is a scalar multiple of $\vec{\mathbf{v}}_1$, and all of these vectors $\vec{\mathbf{w}}$ lie on the same line.

# How about the linear span of 3 vectors?

**Cindy:** I see. And when we take the span *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$ of 3 vectors in $\mathbb{R}^3$, and one of them is a scalar multiple of another, then we can get at most a plane, but not the entire space $\mathbb{R}^3$, right?

**Bob:** Right! Essentially the same calculation shows this.

**Denny:** And when neither of these 3 vectors is a scalar multiple of another, we get the entire $\mathbb{R}^3$!

**Theo:** You are jumping to conclusions here, Denny.

**Denny:** How is that?

**Theo:** Consider $\vec{\mathbf{v}}_1 = [1, 2, 3], \vec{\mathbf{v}}_2 = [-1, 1, -2], \vec{\mathbf{v}}_3 = [3, 0, 7]$.

Neither of these vectors is a scalar multiple of the other two.

**Denny:** So *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3) = \mathbb{R}^3$; that's what I meant!

**Theo:** But that's not the case!

**Bob:** I don't see why.

# What if one vector is a linear combination of the others?

**Theo:** Here $\vec{\mathbf{v}}_3 = [3, 0, 7] = [1, 2, 3] - 2[-1, 1, -2] = \vec{\mathbf{v}}_1 - 2\vec{\mathbf{v}}_2$.

**Bob:** Oh, now I see! When $\vec{\mathbf{w}}$ is in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$, for the vectors in your example, then there are scalars $d_1, d_2, d_3$ such that

$$\begin{aligned}\vec{\mathbf{w}} &= d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + d_3\vec{\mathbf{v}}_3 = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + d_3(\vec{\mathbf{v}}_1 - 2\vec{\mathbf{v}}_2) \\ &= (d_1 + d_3)\vec{\mathbf{v}}_1 + (d_2 - 2d_3)\vec{\mathbf{v}}_2.\end{aligned}$$

So $\vec{\mathbf{w}}$ must already be in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$, which cannot be all of $\mathbb{R}^3$, because it is the linear span of only 2 vectors!

**Theo:** And this observation generalizes: When we have 3 vectors in $\mathbb{R}^3$, one of which is a linear combination of the others, then the linear span cannot be all of $\mathbb{R}^3$. It must have dimension less than 3, like a plane or a line. I can show you the proof!

**Bob:** Thank you Theo, but maybe not now.

## When does a linear span have dimension $k$?

**Cindy:** Would it then be true in general that the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k)$ of $k$ vectors in some $\mathbb{R}^n$ has dimension less than $k$ if one of these vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k$ is a linear combination of the others, and has dimension $k$ if none of these vectors is a linear combination of the others?

**Alice:** Yes, Cindy! What you have expressed here in your own words is a mathematical theorem.

**Denny:** Wow! But look at Cindy's sentence. It goes on and on. Is there a simpler way of saying this; one that I could memorize?

**Frank:** And what do you mean by "dimension," Cindy?

**Cindy:** I simply mean: Lines have dimension 1, planes have dimension 2, and $\mathbb{R}^3$ has dimension 3.

**Frank:** I know that. But what does "dimension 24" mean?

**Theo:** Good points, Denny and Frank! To address your concerns, we will need to introduce some new terminology.

# Linear (in)dependence

**Definition (Tentative)**

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of $k > 1$ vectors of the same order.

This set is *linearly dependent* if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.

This set is *linearly independent* if, and only if, it is not linearly dependent.

**Cindy:** "Of the same order" here means that they are all row vectors or all column vectors, and are all of the same length, right?

**Theo:** Right.

**Cindy:** So, could I now put my observation like this?

"The linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k)$ of $k > 1$ vectors in some $\mathbb{R}^n$ has dimension less than $k$ if these vectors are linearly dependent, and has dimension $k$ if they are linearly independent."

**Question C25.3:** Did Cindy get this right?

**Theo:** Perfect!

**Denny:** And that's a much shorter sentence!

# What did we learn from this conversation?

**Question C25.4:** How would you describe the take-home message from this part of the conversation?