Conversation 25B: Introduction to Linear Dependence and Linear Independence, Part II

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MATH 3200: Applied Linear Algebra

Review: When does a linear span have dimension k?

Cindy: Would it then be true in general that the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k)$ of k vectors in some $\mathbb{R}^n$ has dimension less than k if one of these vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k$ is a linear combination of the others, and has dimension k if none of these vectors is a linear combination of the others?

Alice: Yes, Cindy! What you have expressed here in your own words is a mathematical theorem.

Denny: Wow! But look at Cindy's sentence. It goes on and on. Is there a simpler way of saying this that I could memorize?

Frank: And what do you mean by "dimension," Cindy?

Cindy: I simply mean: Lines have dimension 1, planes have dimension 2, and $\mathbb{R}^3$ has dimension 3.

Frank: I know that. But what does "dimension 24" mean?

Theo: Good points, Denny and Frank! To address your concerns, we will need to introduce some new terminology.

Review: Tentative definition of Linear (in)dependence

Definition (Tentative)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of $k > 1$ vectors of the same order.

This set is *linearly dependent* if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.

This set is *linearly independent* if, and only if, it is not linearly dependent.

Cindy: "Of the same order" here means that they are all row vectors or all column vectors, and are all of the same length, right?

Theo: Right.

Cindy: So could I now put my observation like this?

"The linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k)$ of $k > 1$ vectors in some $\mathbb{R}^n$ has dimension less than k if these vectors are linearly dependent, and has dimension k if they are linearly independent."

Theo: Perfect!

Denny: And that's a much shorter sentence!

Linear (in)dependence: Example 1

Definition (Tentative)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ of $k > 1$ vectors of the same order.

This set is *linearly dependent* if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.

This set is *linearly independent* if, and only if, it is not linearly dependent.

Frank: But this definition is too abstract! Can we look at some examples?

Alice: Let $\vec{\mathbf{v}}_1 = [2, -1, 3]$, $\vec{\mathbf{v}}_2 = [3, 4, 4]$, $\vec{\mathbf{v}}_3 = [6, -3, 9]$.

Question C25.5: Is the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ linearly dependent or linearly independent?

Frank: Linearly dependent. This is like in my counterexample for Question 6 of Module 43. Here

$\vec{\mathbf{v}}_3 = 3\vec{\mathbf{v}}_1 = 3\vec{\mathbf{v}}_1 + 0\vec{\mathbf{v}}_2$,

so $\vec{\mathbf{v}}_3$ is a linear combination of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

Linear (in)dependence: Example 2

Definition (Tentative)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ of $k > 1$ vectors of the same order.

This set is *linearly dependent* if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.

This set is *linearly independent* if, and only if, it is not linearly dependent.

Alice: Let $\vec{\mathbf{v}}_1 = [2, -1, 3]$, $\vec{\mathbf{v}}_2 = [0, 0, 0]$, $\vec{\mathbf{v}}_3 = [6, -3, 8]$.

Question C25.6: Is the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ linearly dependent or linearly independent?

Cindy: Linearly dependent. Here
$\vec{\mathbf{v}}_2 = \vec{\mathbf{0}} = 0\vec{\mathbf{v}}_1 + 0\vec{\mathbf{v}}_3$ is a linear combination of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_3$.
Actually, this example shows that when one of the vectors in your set is a zero vector $\vec{\mathbf{0}}$, then this set must always be linearly dependent.

Bob: But what if your set of vectors is $\{\vec{\mathbf{0}}\}$ and contains only the zero vector? Our definition doesn't say, because it assumes that $k > 1$ so that we have at least two vectors.

Linear (in)dependence: The official definition

Theo: Here is the official definition that is in all the textbooks:

Definition (Official)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of vectors of the same order.

This set is *linearly dependent* if, and only if, there are scalars $c_1, c_2, \ldots, c_k$, not all of them zero, so that
$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

This set is *linearly independent* if, and only if, it is not linearly dependent.

Bob: Oh, now I see! If your set of vectors is $\{\vec{\mathbf{v}}_1\} = \{\vec{\mathbf{0}}\}$, then we can take any nonzero scalar c_1, for example, $c_1 = 2021$, and we get $c_1\vec{\mathbf{v}}_1 = 2021\vec{\mathbf{0}} = \vec{\mathbf{0}}$. So the set is linearly dependent, and Cindy's observation also works for this case!

Cindy: So, our tentative definition was wrong and we should use the official one? I'm all confused.

For $k > 1$ the two definitions are equivalent

Alice: There is nothing wrong with our tentative definition. For $k > 1$ the two definitions are *equivalent,* which means they describe the same property. But our tentative definition was easier to understand.

Frank: I don't believe that they are equivalent.

Alice: Good. So let's verify it then.

Theo: I'll be happy to show you a formal proof.

Bob: Can we look at some numerical example first?

Alice: Suppose $\vec{\mathbf{v}}_1 = 2\vec{\mathbf{v}}_2 - 3\vec{\mathbf{v}}_3 + 0\vec{\mathbf{v}}_4$ so that set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is linearly dependent in the sense of our tentative definition.

To show that this set is also linearly dependent in the sense of the official definition, we need coefficients c_1, c_2, c_3, c_4 such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 + c_4\vec{\mathbf{v}}_4 = \vec{\mathbf{0}}$.

Denny: Could we take $c_2 = 2, c_3 = -3, c_4 = 0$?

Alice: This will work, but we still need c_1.

Question C25.7: What choice of c_1 would work with Denny's choice of the other coefficients?

For $k > 1$ the two definitions are equivalent, continued

Alice: We need $c_1 = -1$ here.

We have observed that if $\vec{\mathbf{v}}_1 = 2\vec{\mathbf{v}}_2 - 3\vec{\mathbf{v}}_3 + 0\vec{\mathbf{v}}_4$, so that set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is linearly dependent in the sense of our tentative definition,

then $\quad -\vec{\mathbf{v}}_1 + 2\vec{\mathbf{v}}_2 - 3\vec{\mathbf{v}}_3 + 0\vec{\mathbf{v}}_4 = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 + c_4\vec{\mathbf{v}}_4 = \vec{\mathbf{0}}.$

This shows linear dependence in the sense of the official definition, since not all of our coefficients $c_1 = -1, c_2 = 2, c_3 = -3, c_4 = 0$ are zero.

Frank: But doesn't "equivalent" also mean that if there are at least two vectors and they are linearly dependent according to the official definition, they also have to be linearly dependent according to our tentative one?

Alice: Very important point, Frank!

So let's look at an example of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ such that

$0\vec{\mathbf{v}}_1 + 5\vec{\mathbf{v}}_2 + 4\vec{\mathbf{v}}_3 - 0.2\vec{\mathbf{v}}_4 = \vec{\mathbf{0}}.$

They are linearly dependent in the sense of the official definition.

Question C25.8: How can we express one of these vectors as a linear combination of the other vectors?

Alice: For example, $\vec{\mathbf{v}}_4 = 0\vec{\mathbf{v}}_1 + 25\vec{\mathbf{v}}_2 + 20\vec{\mathbf{v}}_3$.

Linear (in)dependence: Example 3

Alice: Let $\vec{\mathbf{v}}_1 = [2, -1, 3]$, $\vec{\mathbf{v}}_2 = [3, 4, 5]$, $\vec{\mathbf{v}}_3 = [6, -7, 8]$.

Question: Is the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ linearly dependent or linearly independent?

Cindy: I don't know—

Frank: Looks linearly independent to me.

Denny: Maybe linearly dependent, but I can't see how—

Bob: Hard to tell. I guess I would need to check for each of the three vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ whether it is a linear combination of the other two vectors. For that we would need to solve three systems of linear equations.

Denny: Way too much work, Bob! There must be a shortcut.

Bob: I was thinking of using our tentative definition. But perhaps the calculations simplify when we use the official one.

Linear (in)dependence: Example 3, continued

Definition (Official)

A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of vectors of the same order is *linearly dependent* if, and only if, there are scalars $c_1, c_2, \ldots, c_k$, not all of them zero, so that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

This set is *linearly independent* if, and only if, it is not linearly dependent.

Alice: We want to find all vectors of coefficients c_1, c_2, c_3 with $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = c_1[2, -1, 3] + c_2[3, 4, 5] + c_3[6, -7, 8] = \vec{\mathbf{0}}$.

These must be the solutions of the linear system

$$\begin{array}{rcrcrcl} 2c_1 & + & 3c_2 & + & 6c_3 & = & 0 \\ -c_1 & + & 4c_2 & - & 7c_3 & = & 0 \\ 3c_1 & + & 5c_2 & + & 8c_3 & = & 0 \end{array}$$

Since this system is homogeneous, the vector $[c_1, c_2, c_3]^T = [0, 0, 0]^T$ is always a solution.

Question C25.9: What can you say about linear (in)dependence of the above set of vectors if this system has more than one solution?

Linear (in)dependence: Example 3, towards completion

If $[0, 0, 0]^T$ is the only solution of the system, then these three vectors are linearly independent; if the system is underdetermined so that there are other solutions, the vectors are linearly dependent.

Cindy: So, now we would need to solve the system

$$\begin{array}{rcrcrcl} 2c_1 & + & 3c_2 & + & 6c_3 & = & 0 \\ -c_1 & + & 4c_2 & - & 7c_3 & = & 0 \\ 3c_1 & + & 5c_2 & + & 8c_3 & = & 0 \end{array}$$

by Gaussian elimination?

Alice: Yes, this would be one good method.

Denny: Can we leave this for later?

Bob: In fact, I can see here that we are supposed to do this in Module 47.

Take-home message: Linear (in)dependence

A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ of vectors of the same order is *linearly dependent* if, and only if, there are scalars $c_1, c_2, \dots, c_k$, not all of them zero, so that

$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \dots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

The set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is *linearly independent* if, and only if, it is not linearly dependent.

When $k > 1$, the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is linearly dependent if, and only if, one of these vectors can be expressed as a linear combination of the others.