Conversation 25C: A Geometric Interpretation of Linear (In)dependence

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MATH 3200: Applied Linear Algebra

Review: Definitions of Linear (in)dependence

Definition (Tentative)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of vectors of the same order.

This set is *linearly dependent* if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.

This set is *linearly independent* if, and only if, it is not linearly dependent.

Definition (Official)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of vectors of the same order.

This set is *linearly dependent* if, and only if, there are scalars $c_1, c_2, \ldots, c_k$, not all of them zero, so that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

This set is *linearly independent* if, and only if, it is not linearly dependent.

These two definitions are equivalent; they define the same concepts.

Review: Two observations

Recall the following fact from Module 46:

Proposition

Consider a set $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}\}$ *of vectors of the same order.*

(a) *Every vector* $\vec{\mathbf{w}}$ *in span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$ *is also in span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.

(b) *If* $\vec{\mathbf{v}}_{k+1}$ *is in span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$, *then span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k) =$ *span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.

Also recall Cindy's observation from Conversation 25:

Proposition

The linear span span$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k)$ *of* $k > 1$ *vectors in some* $\mathbb{R}^n$ *has dimension less than k if these vectors are linearly dependent, and has dimension k if they are linearly independent.*

How to think of "linear independence"?

Denny: How can I wrap my mind around this "linearly independent" thing? The definitions only say "when it's not linearly dependent." Is there some positive way of thinking about linear independence?

Alice: Yes, there is. Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be a linearly independent set of vectors in some space $\mathbb{R}^n$. For starters, let's focus on the subset $\{\vec{\mathbf{v}}_1\}$ that consists only of the first of these vectors.

Question C25.10: Is the set $\{\vec{\mathbf{v}}_1\}$ also linearly independent?

Denny: Yes, because any subset of a linearly independent set is linearly independent.

Question C25.11: Geometrically speaking, what would $span(\vec{\mathbf{v}}_1)$ then be?

Denny: By Cindy's observation, this set must be a 1-dimensional linear subspace of $\mathbb{R}^n$, a line through the origin.

Linear independence: The geometric intuition

Alice: Right! Now consider the subset $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ that consists only of the first two vectors. Then $\vec{\mathbf{v}}_1$ is in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$.

Question C25.12: Can $\vec{\mathbf{v}}_2$ also be in $span(\vec{\mathbf{v}}_1)$?

Denny: No. This is ruled out by our tentative definition, since $\vec{\mathbf{v}}_2$ cannot be a linear combination of any of the other vectors in the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ if this set is linearly independent.

Alice: So, $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ must be a **larger** set than $span(\vec{\mathbf{v}}_1)$.

Denny: I see. By Cindy's observation, it would be a 2-dimensional set, a plane through the origin!

Alice: Right! Analogously, $\vec{\mathbf{v}}_3$ is not in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$, so that $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$ must be still larger than the plane $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$; it must be of dimension 3.

Denny: And so on. Got it. Are you saying that in a linearly independent set, each vector adds a new dimension to the span?

Alice: Yes, this is how I think about it geometrically.

Take-home message

This discussion shows how we can think geometrically about *linear independence.*

Question C25.13: How would you describe, in your own words, the geometric interpretation of *linear dependence*?

We can think about linear dependence in this way: When the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is linearly dependent, and we start from the zero-dimensional space $V = \{\vec{\mathbf{0}}\}$ and then successively form the sets *span*$(\vec{\mathbf{v}}_1)$, *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$, *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$, then at some step the dimension does *not* increase.