# Conversation 26: Bases and Dimension

Winfried Just
Department of Mathematics, Ohio University

MATH 3200: Applied Linear Algebra

# Review: Definitions of linear (in)dependence

## Definition (Tentative)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ of vectors of the same order.

This set is *linearly dependent* if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.

This set is *linearly independent* if, and only if, it is not linearly dependent.

## Definition (Official)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ of vectors of the same order.

This set is *linearly dependent* if, and only if, there are scalars $c_1, c_2, \dots, c_k$, not all of them zero, so that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

This set is *linearly independent* if, and only if, it is not linearly dependent.

These two definitions are equivalent; they define the same concepts.

# Review: Two observations

Recall the following fact from Module 46:

**Proposition**

*Consider a set* $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}\}$ *of vectors of the same order.*

(a) *Every vector* $\vec{\mathbf{w}}$ *in span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$ *is also in span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.

(b) *If* $\vec{\mathbf{v}}_{k+1}$ *is in span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$, *then span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k) =$ *span*$(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.

Also recall Cindy's observation from Conversation 25:

**Proposition**

*The linear span span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k)$ *of* $k > 1$ *vectors in some* $\mathbb{R}^n$ *has dimension less than k if these vectors are linearly dependent, and has dimension k if they are linearly independent.*

# The vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n$

**Alice:** Consider the following vectors in $\mathbb{R}^n$ for a given $n$:

$$\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots \qquad \vec{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

**Theo:** These are called the *standard basis vectors.*

**Frank:** What's that supposed to mean?
Are there also nonstandard basis vectors?

**Alice:** We will come to that soon.
For now, let us just note that they are linearly independent:
If $n = 3$, you can think of $span(\vec{\mathbf{e}}_1)$ as the $x$-axis, of $span(\vec{\mathbf{e}}_2)$ as the $y$-axis, of $span(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ as the $x$-$y$-plane, and of $span(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)$ as the entire 3-dimensional space $\mathbb{R}^3$. So each of these vectors takes us into a new dimension, as Denny had observed.

# Moving backwards

**Frank:** You are talking about dimension again, but we still don't even have defined what this means!

**Alice:** So, Frank, how would you think about linear dependence and linear independence without referring to "dimension?"

**Frank:** From the opposite angle. Let $V$ be a vector space, and let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ be a set of vectors such that $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$.

I forgot the official name for such a set $S$; can you remind us, Theo?

**Theo:** Such a set $S$ is called spanning set for $V$.

**Frank:** When the set $S$ is linearly dependent, then one of these vectors is a linear combination of the others, and you can throw it out, so that you end up with a smaller spanning set $S^-$ for the same space $V$.

**Cindy:** I'm not sure why. Can you explain this, please?

# Why can we remove one element of the spanning set?

**Alice:** Good question, Cindy!
We always need to carefully check each step in a reasoning.

**Question C26.1:** Which fact that we have seen earlier shows that this is possible?

**Theo:** The proposition from Module 46 says:

(b) If $\vec{\mathbf{v}}_{k+1}$ is in $span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$, then
$span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k) = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.

So when $\vec{\mathbf{v}}_{k+1}$ is a linear combination of the other vectors, we can remove $\vec{\mathbf{v}}_{k+1}$ and still get a spanning set for the same vector space.

**Bob:** But wait! What if, for example, $\vec{\mathbf{v}}_1$ is a linear combination of the other vectors, but $\vec{\mathbf{v}}_{k+1}$ is not?

**Theo:** Precisely! Then the proposition would not formally apply.

**Alice:** But then we can change the numbering of the spanning set so that $\vec{\mathbf{v}}_1$ gets listed last and apply the proposition to the same set, but with the new numbering.

**Frank:** Right! I am only talking about removing elements from a set; you can number the elements in any way you want. The particular numbering is only needed to make a connection with the fact that we learned in Module 46.

Now, if $S^-$ is still linearly dependent, we can remove another element from it without changing the linear span.

And so on.

**Denny:** Will this removing and removing ever stop?

**Frank:** Yes. Eventually we must be left with a set where none of the vectors is a linear combination of the others. It will have the same linear span as the set we started with, but will be linearly independent by our tentative definition. I'd call it $B$, because it is so basic and contains the bare minimum of what we need in terms of a spanning set for $V$.

**Theo:** Excellent choice of letters, Frank! Such a linearly independent spanning set is actually called a *basis* of $V$. Here is the official definition:

**Definition**

Let $V$ be a vector space. A linearly independent spanning set of $V$ is called a *basis* of $V$.

**Alice:** Frank's argument essentially is a proof of the following:

**Theorem**

*Let $V = span(S)$ for some set of vectors $S$. Then $S$ contains a subset $B$ that is a basis of $V$.*

**Cindy:** I'm still confused. Can you give us a concrete example, Frank?

# An example of a basis of a vector space

**Frank:** Let $S = \{[2,-2,0],[1,1,0],[3,-1,0],[4,4,0]\}$, and let $V = span(S)$.

**Denny:** Since $[4,4,0] = 4[1,1,0]$, we can kick out $[4,4,0]$ and get $S^- = \{[2,-2,0],[1,1,0],[3,-1,0]\}$. Then $V = span(S^-)$.

**Bob:** Next we can remove $[3,-1,0]$, since
$[3,-1,0] = [2,-2,0] + [1,1,0]$.
So we get $B = \{[2,-2,0],[1,1,0]\}$.
Then $V = span(B) = span([2,-2,0],[1,1,0])$.

**Question C26.2:** Is this set $B$ is linearly independent?

**Alice:** Yes. A set of two vectors can be linearly dependent only if one vector is a scalar multiple of the other, which is not the case for the vectors in $B$.

**Frank:** Now we can see that $B$ is a basis of $V = span(S)$.

# Another option?

Consider $S = \{[2, -2, 0], [1, 1, 0], [3, -1, 0], [4, 4, 0]\}$,
and let $V = span(S)$.

**Cindy:** But couldn't we also go like this:

1. Remove $[2, -2, 0]$, because this is equal to $[3, -1, 0] - [1, 1, 0]$.
2. Remove $[1, 1, 0]$, because it is equal to $0.25[4, 4, 0]$.
3. Argue that the remainder $B_1 = \{[3, -1, 0], [4, 4, 0]\}$ is linearly independent?

**Frank:** Sure, why not?

**Cindy:** But which set is the basis of $V$ then? Your $B$ or my $B_1$?

**Question C26.3:** How would you respond to Cindy here?

**Bob:** Both sets $B$ and $B_1$ are bases for $V$.

# Multiple bases for the same space

**Definition**

Let $V$ be a vector space. A linearly independent spanning set of $V$ is called *a basis* of $V$.

**Theo:** According to this definition, a given vector space will usually have many different bases. In our example, both $B$ and $B_1$ are bases for $V = span([2, -2, 0], [1, 1, 0], [3, -1, 0], [4, 4, 0])$.
And there are many more, for example $B_2 = \{[1, 1, 0], [3, -1, 0]\}$, $B_3 = \{[2, -2, 0], [3, -1, 0]\}$, $B_4 = \{[1, 0, 0], [0, 1, 0]\}$, ...

**Denny:** I don't buy that last example.
Your set $B_4$ isn't even contained in the original spanning set!

**Theo:** It doesn't have to be. There are usually many different spanning sets for the same vector space $V$. Here one can show that $V$ is simply the $x$-$y$-plane, and $B_4$ is the set of the standard basis vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ that Alice had mentioned if we work with row instead of column vectors.

**Frank:** Cool! So $B_4$ would be a standard basis, and the other ones would be "nonstandard" bases for the same vector space.

# The size of bases

**Bob:** The bases of $V$ that we found and the ones that you mentioned all have the same size; each contains 2 vectors. Can you give us examples of bases of different sizes for this space?

**Theo:** No, I can't. All bases of a given $V$ have the same size.

**Theorem**

*Let $V$ be any vector space and let $B_1, B_2$ be two bases of $V$. Then $B_1$ and $B_2$ have the same size.*

**Denny:** Interesting! Why is that?

**Theo:** The proof is a little complicated and our Professor doesn't want to cover it in the course, but I will be happy to show it to you.

**Denny:** Thank you, Theo, thank you **very much.** Let's skip it.

**Cindy:** So for our $V$, which is a plane, all these bases for our $V$ have size 2, which is equal to the dimension of $V$, right?

# Bases and dimension

**Frank:** You are talking about "dimension" again. But I still don't know what this actually means, at least not for dimensions larger than 3.

**Theo:** Here is the formal definition for you:

**Definition**

Let $V$ be any vector space. Then the *dimension* of $V$, denoted by $dim(V)$, is the size of any basis of $V$.

This definition is unambiguous because any two bases of $V$ have the same size.

**Alice:** What Cindy was noticing here is that this definition agrees with our geometric intuition about dimension for the familiar dimensions 1, 2, and 3.

**Frank:** OK, thank you! That clarifies it.

# Take-home message: Bases of vector spaces

Let $V$ be a vector space. A set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ such that $V = span(S) = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$ is called a *spanning set* of $V$.

A linearly independent spanning set of $V$ is called a *basis* of $V$.

Every spanning set $S$ of a vector space $V$ contains a basis $B$.

A vector space has usually many different bases, but every two bases for the same vector space $V$ have the same size. This size is called the *dimension* of $V$ and denoted by *$dim(V)$.*

For a given $n$, we let $\vec{\mathbf{e}}_i$ denote the vector in $\mathbb{R}^n$ that has 1 in position $i$ and 0 in all other positions. The set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \dots, \vec{\mathbf{e}}_n\}$ forms the *standard basis* of $\mathbb{R}^n$.
Its elements $\vec{\mathbf{e}}_i$ are called *standard basis vectors.*