Conversation 27: An Application of Base Changes

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MATH 3200: Applied Linear Algebra
Spring Semester 2019

What are base changes good for?

Frank: Why would anywone ever want to consider a nonstandard basis for $\mathbb{R}^n$ and alternative coordinates?

Theo: We have seen in Lecture 25 that when we are interested in a low-dimensional subspace V of a high-dimensional space $\mathbb{R}^n$, then we could use a basis B for V to parametrize the vectors in V with very few numbers. Unless V is the linear span of some coordinate axes, this B would need to be a nonstandard basis.

Frank: I got that. But I am talking about nonstandard bases for an entire space $\mathbb{R}^n$. Why would anyone want to use those?

Alice: Such nonstandard bases are used even in everyday household applications.

Frank: No way!

Alice: Remember Marilyn and Marvin who went on a diet?

Review: A health-conscious couple

They decided to go on Dr. Losit's new scientifically proven diet and purchased three products from his company: Losit-Quick (strawberry taste, which they both like), Losit-Fast (Marylin's beloved watermelon taste that Marvin cannot stand), and Losit-Easy (beer-flavored for Marvin, detested by Marylin).

- One serving of Losit-Quick contains 20 grams of sugar, 200 grams of protein, and 20 grams of fat.
- One serving of Losit-Fast contains 15 grams of sugar, 50 grams of protein, and 40 grams of fat.
- One serving of Losit-Easy contains 25 grams of sugar, 150 grams of protein, and 60 grams of fat.

Their problem was how to compose from these ingredients a meal that contains exactly the recommended amounts of sugar, protein, and fat.

The Cartesian coordinates of the ingredients and the meal

The meal and the contents of its ingredients can be represented here as column vectors in $\mathbb{R}^3$, where the first coordinate specifies the amount of sugar (in grams), the second coordinate the amount of protein (in grams), and the third coordinate the amount of fat (in grams).

Cindy: Would this be the Cartesian coordinates in this problem?

Alice: Yes, this is how I think about it. Let $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F, \vec{\mathbf{v}}_E, \vec{\mathbf{v}}_M$ be the vectors of grams of sugar, protein, and fat in one serving of Losit-Quick, Losit-Fast, Losit-Easy, and in the meal that was recommended by Dr. Losit.

$$\vec{\mathbf{v}}_Q = \begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} \qquad \vec{\mathbf{v}}_F = \begin{bmatrix} 15 \\ 50 \\ 40 \end{bmatrix} \qquad \vec{\mathbf{v}}_E = \begin{bmatrix} 25 \\ 150 \\ 60 \end{bmatrix} \qquad \vec{\mathbf{v}}_M = \begin{bmatrix} 50 \\ 300 \\ 100 \end{bmatrix}$$

In order to compose the meal, our protagonists needed to find coefficients c_Q, c_F, c_E that represent the amounts of servings of each ingredient so that $c_Q\vec{\mathbf{v}}_Q + c_F\vec{\mathbf{v}}_F + c_E\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M$.

Our alternative coordinates: Numbers of Servings

Bob: So these numbers of servings would be the alternative coordinates here? This would make sense, because if I cook a meal, I think about how much of each ingredient to add, not about how much fat, protein, and sugar each ingredient has.

Alice: This is why I call this an everyday household application.

Cindy: We need to specify a basis for the alternative coordinates. We should say that the numbers of servings would be alternative coordinates with respect to the basis $B = \{\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F, \vec{\mathbf{v}}_E\}$.

Denny: But I recall that poor Marvin couldn't compose the meal he wanted, so how can this be basis?

Alice: Marvin could not compose the meal as a linear combination of only his favorite ingredients $\vec{\mathbf{v}}_Q$ and $\vec{\mathbf{v}}_E$.

Question C27.1: What does this imply about $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E)$?

This implies that $\vec{\mathbf{v}}_M$ is not in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E)$.

The linear span of B

Alice: Also notice that neither of the vectors $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E$ is a scalar multiple of the other.

Question C27.2: What does this imply about $dim(span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E))$?

It implies that the set $\{\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E\}$ is linearly independent so that $dim(span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E)) = 2$.

Alice: On the other hand, Marilyn could compose the meal from her favorite ingredients $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F$. This means that $\vec{\mathbf{v}}_M$ is in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F)$, and therefore $\vec{\mathbf{v}}_M$ is in $span(B)$. So $span(B)$ must be a larger set than $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E)$ and must have dimension 3.

Denny: Oh, I see! So $span(B)$ must be the entire 3-dimensional space $\mathbb{R}^3$, and since B has only 3 elements, it must be a minimal spanning set for $\mathbb{R}^3$, which means a basis for $\mathbb{R}^3$.

Question C27.3: Did Denny get this right?

Yes.

How about consulting a different specialist?

Question C27.4: Marilyn found that $\vec{\mathbf{v}}_Q + 2\vec{\mathbf{v}}_F = \vec{\mathbf{v}}_M$. What does this imply about the alternative coordinates $\vec{\mathbf{c}}_M$ with respect to B of Dr. Losit's meal?

The alternative coordinates are $\vec{\mathbf{c}}_M = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Bob: So Marvin was out of luck because $\vec{\mathbf{v}}_M$ is already in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F)$ and he could not add any of his beer-flavored ingredient, because B is a basis and each vector in $\mathbb{R}^3$ has unique alternative coordinates.

Frank: This guy Dr. Losit is clearly a quack. Marvin should have consulted a different specialist who recommends a meal $\vec{\mathbf{v}}_D$ that is *not* in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F)$. Then the amount of servings of $\vec{\mathbf{v}}_E$ would be nonzero.

And if, moreover, $\vec{\mathbf{v}}_D$ is in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E)$, Marvin wouldn't even have to ingest any of the stuff with the water melon taste!

Would this make Marvin happy?

Theo: This is correct. When $\vec{\mathbf{v}}_D$ is not in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F)$, then we can have $\vec{\mathbf{v}}_D = c_1\vec{\mathbf{v}}_Q + c_2\vec{\mathbf{v}}_F + c_3\vec{\mathbf{v}}_E$ only if $c_3 \neq 0$. Similarly, if $\vec{\mathbf{v}}_D$ is in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E)$, then $\vec{\mathbf{v}}_D = c_1\vec{\mathbf{v}}_Q + c_3\vec{\mathbf{v}}_E = c_1\vec{\mathbf{v}}_Q + c_2\vec{\mathbf{v}}_F + c_3\vec{\mathbf{v}}_E$ only if $c_2 = 0$.

Denny: So if $\vec{\mathbf{v}}_D$ is both in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E)$ and not in $span(\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F)$, then Marvin will be perfectly happy with his meal!

Question C27.5: Do you agree with Denny?

Not necessarily. Consider, for example,

$$\vec{\mathbf{v}}_D = \begin{bmatrix} 55 \\ 650 \\ 20 \end{bmatrix} = 4\begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} - \begin{bmatrix} 25 \\ 150 \\ 60 \end{bmatrix} = 4\vec{\mathbf{v}}_Q - \vec{\mathbf{v}}_E$$

Here Marvin would need to use four servings of Losit-Quick and minus one seving of his favorite Losit-Easy. Not clear how he would do this, but it is hardly a recipe for happiness ...