Conversation 2: Our First Close Encounter with Proofs

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MATH 3200: Applied Linear Algebra

Review: The instructor's gradebook

During this semester, your instructor will record the grades of all students in this class in a matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

Here m is the number of students in the class.

The entries $a_{11}, a_{21}, \dots, a_{m1}$ in the first column will normally be the names of the students, but we will assume here that your instructor does not record the names.

Then n will be the number of graded items, which we will think of here as tests or quizzes, and the entry a_{ij} represents the score of student i on gradable item number j.

How to calculate the mean overall score?

Cindy: I'm wondering: How would one calculate the mean overall score for the entire class from this matrix?

Bob: Add up the total scores of all students and divide the sum by the number m of rows.

Denny: No!!!

You need to add up the mean scores of all graded items.

Cindy: Perhaps. But Denny, should we then divide the total score for each item by the total number of students m, or, for each of the tests and quizzes, by the number of students who actually took the test or quiz?

Question C2.1: What denominator should we use in Denny's method?

Both methods give the same result. Do they?

Bob: Here we are interested in mean overall scores, not in mean performance of students who took a test or quiz.

So Denny would need to divide by m for each item.

Then Denny's method will give the exact same result as mine.

Denny: Not always. My method is the correct one. Believe me.

Bob: The results of both calculations will always be the same.

Cindy: Can you prove this, Bob?

Frank: Come on Cindy! This is a course for engineering students. Who needs proofs?

Cindy: I'm not sure whom to believe, Bob or Denny.

If I saw a proof, I would know.

Both methods give the same result. Do they?

Frank: OK, OK! Here is your proof:

Let $\mathbf{A} = \begin{bmatrix} 65 & 92 \\ 45 & 78 \end{bmatrix}$

In Bob's method, we add up each row, then add up these sums, and then divide by 2.

This gives: $((65 + 92) + (45 + 78))/2 = 140.$

In Denny's method, we compute the averages for each column and add them up.

This gives: $(65 + 45)/2 + (92 + 78)/2 = 140.$

We get the same number, so Bob is right.

Question C2.2: Has Frank proved that Bob is right?

What's a proof?

Theo: But that's not a proof.

Frank: Why not?

Bob: Because I said "*Always* give the same result."

Denny: And I am telling you: "*Not always.*"

Cindy: You have shown us **only one example** where we get the same number, Frank. This doesn't tell me whom to believe, Bob or Denny.

Frank: But one cannot do the calculations for all possible examples!

Theo: Yes one can.

Frank: Get real, man! There are infinitely many possible examples. Not even a computer could do that many calculations in a finite amount of time!

Alice: Good point, Frank!

Proofs are calculations with symbols

Frank: As I said: Forget about these "proofs."

Theo: In a proof, one needs to *do the calculations with symbols, not with particular numbers.*

Cindy: Like, how??

Bob: I think Theo wants us to write like this: Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

In my method, we add up each row, then add up these sums, and then divide by 2.
This gives $((a+b)+(c+d))/2 = (a+b+c+d)/2$.

In Denny's method, we compute the averages for each column and add them up. This gives
$(a+c)/2 + (b+d)/2 = (a+b+c+d)/2$; the same result.

Question C2.3: Has Bob proved his claim?

How to prove it for all matrices?

Cindy: I can see now that the methods give the same result for all 2×2 matrices. But what if the matrix is of order 2×3?

Denny: I am telling you: "*Not always.*"

Bob: Then let $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

In this case my method ...

Cindy: And what if the matrix is of order 10×10?

Bob: Wait, I'm running out of letters ...

Cindy: Or of order $100 \times 10,000$? Or ...

Frank: Which brings us back to my point.

Question C2.4: Now what do we do?

Alice comes to our rescue

Alice: For a proof that works for matrices of all orders, we need to assume that **A** is of order $m \times n$ for some positive integers m and n, where m and n are written as symbols instead of specific numbers.

We would start with: Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

In Bob's method we sum up each row, then we take the sums of these sums, and divide by m. This gives:
$((a_{11} + a_{12} + \dots + a_{1n}) + (a_{21} + a_{22} + \dots + a_{2n}) + \dots$
$\dots + (a_{m1} + a_{m2} + \dots + a_{mn}))/m.$

In Denny's method we sum up each column, divide by m, and then we take the sums of these terms. This gives:
$(a_{11} + a_{21} + \dots + a_{m1})/m + (a_{12} + a_{22} + \dots + a_{m2})/m + \dots$
$\dots + (a_{1n} + a_{2n} + \dots + a_{mn})/m.$

Alice completes the proof

Alice: Let us slightly rewrite these expressions by dropping brackets and distributing the factor $1/m$:

$$((a_{11} + a_{12} + \cdots + a_{1n}) + (a_{21} + a_{22} + \cdots + a_{2n}) + \dots$$
$$\cdots + (a_{m1} + a_{m2} + \cdots + a_{mn}))/m$$
$$= a_{11}/m + a_{12}/m + \cdots + a_{1n}/m + a_{21}/m + a_{22}/m + \cdots + a_{2n}/m + \dots$$
$$\cdots + a_{m1}/m + a_{m2}/m + \cdots + a_{mn}/m.$$

$$(a_{11} + a_{21} + \cdots + a_{m1})/m + (a_{12} + a_{22} + \cdots + a_{m2})/m + \dots$$
$$+(a_{1n} + a_{2n} + \cdots + a_{mn})/m$$
$$= a_{11}/m + a_{21}/m + \cdots + a_{m1}/m + a_{12}/m + a_{22}/m + \cdots + a_{m2}/m + \dots$$
$$\cdots + a_{1n}/m + a_{2n}/m + \cdots + a_{mn}/m.$$

Thus we get sums of the exact same terms, only added up in a different order. This completes the proof.

Cindy: Wow!! But this looks all very complicated.

Alice: This is our first encounter with proofs.
Over time we will get some practice, and then it will become easier to read proofs and write simple ones ourselves.

Is there a better way to write this proof?

Cindy: Ok. I am now convinced that Bob is right.
But all these "..." make it really, really hard to follow what is going on.

$$((a_{11} + a_{12} + \cdots + a_{1n}) + (a_{21} + a_{22} + \cdots + a_{2n}) + \dots$$
$$\cdots + (a_{m1} + a_{m2} + \cdots + a_{mn}))/m$$
$$= a_{11}/m + a_{12}/m + \cdots + a_{1n}/m + a_{21}/m + a_{22}/m + \cdots + a_{2n}/m + \dots$$
$$\cdots + a_{m1}/m + a_{m2}/m + \cdots + a_{mn}/m.$$

$$(a_{11} + a_{21} + \cdots + a_{m1})/m + (a_{12} + a_{22} + \cdots + a_{m2})/m + \dots$$
$$+(a_{1n} + a_{2n} + \cdots + a_{mn})/m$$
$$= a_{11}/m + a_{21}/m + \cdots + a_{m1}/m + a_{12}/m + a_{22}/m + \cdots + a_{m2}/m + \dots$$
$$\cdots + a_{1n}/m + a_{2n}/m + \cdots + a_{mn}/m.$$

Just looking at these formulas makes my head spin!

Bob: Good point, Cindy!
Perhaps there is a better way of writing up this proof?

Theo: In fact there is. It will become more understandable if we use so-called *Σ-notation* or *summation notation*.
We will learn about this notation in Module 2.

Take-home message

This was our first encounter with proofs in this course. It may have gone a bit fast, but we will practice writing proofs a lot, so you will eventually learn how to do it yourself.

Question C2.5: Describe what you learned about proofs from this conversation.

- Proofs are tools for verifying that mathematical methods, such as recipies for performing certain calculations, always give correct results.
- A single numerical example does not constitute a proof.
- The kind of proofs that we will do in this course are essentially calculations with symbols.
- In order to prove that a given calculation always works as claimed, we need to set up a notation that covers all possibilities.