Conversation 30A: Matrix Representations of Linear Transformations

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MATH3200: Applied Linear Algebra

So many concepts!

Cindy: There were so many concepts covered in the recent lectures: linear combinations, linear span, linear dependence, linear independence, vector spaces, bases, dimension, rank, nullspace, ... I got lost!!

Alice: Yes, this material may look a little overwhelming at first. Let's talk about how these notions are related to each other; this will give us some clarity.

Cindy: Yes, please!!

Denny: You forgot to mention linear transformations, Cindy. I find these most baffling among this whole stuff.

Bob: Let's review the definition of linear transformations from the lecture.

Linear transformations: Our definition

Definition (For the purpose of this course)

Let n, m be positive integers and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then L is called a *linear transformation* if it satisfies both of the following conditions for all vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^n$ and all scalars λ in $\mathbb{R}$:

(i) $L(\lambda\vec{\mathbf{v}}) = \lambda L(\vec{\mathbf{v}})$

(ii) $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$.

Frank: Are they good for anything in mechanical engineering?

Alice: In mechanical engineering, don't you consider what happens if mechanical forces deform objects and move them around?

Frank: We sure do.

Alice: In Lecture 30 we saw some examples of linear transformations of $\mathbb{R}^2$. You can think of them as describing certain ways of moving around a sheet of material and perhaps deforming it in the process. There are similar examples for solid objects in $\mathbb{R}^3$.

Are all linear transformations like this?

Bob: In all of these examples the linear transformation L was of the form $L = L_{\mathbf{A}}$ for some marix $\mathbf{A}$, so that $L_{\mathbf{A}}(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$ for every input vector $\vec{\mathbf{x}}$. These linear transformations were not all that difficult to understand once one understands matrix multiplication.

Denny: Are there any linear transformations that are not of this form? Just curious.

Alice: Yes. Recall our discussions of Markov Chains as simple models of weather forecasting.

Cindy: You mean, like, when we had some probabilities for sunshine and rain today, the model would give us probabilities for sunshine and rain tomorrow?

Alice: Yes. We can look at such a model as a function L that takes probability distributions for the weather on a given day as inputs and *transforms* them into probability distributions for the next day. Such a function L is also a linear transformation.

All linear transformations of column vectors are of this form

Cindy: Didn't we compute this L also by matrix multiplication?

Alice: Yes, but there we multiplied *row vectors* $\vec{\mathbf{x}}$, such as probability distributions, with the matrix $\mathbf{P}$ of transition probabilities to transform them into *row vectors* $\vec{\mathbf{x}}\mathbf{P}$. The order of multiplication of the vector with the matrix was reversed.
So, this linear transformation L is not of the form $L_{\mathbf{A}}$.

Theo: But every linear transformation of *column* vectors will be of the form $L_{\mathbf{A}}$ for some matrix $\mathbf{A}$. There is a theorem about this:

Theorem (Matrix representation of linear transformations)

Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

If both the elements of the domain $\mathbb{R}^n$ of L and the function values $L(\vec{\mathbf{x}})$ in $\mathbb{R}^m$ are treated as column vectors. Then there exists a matrix $\mathbf{A}$ of order $m \times n$ such that $L = L_{\mathbf{A}}$, that is,

$L(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}}$ in $\mathbb{R}^n$.

How to find the matrix?

Denny: Why would this be true, Theo?

Frank: Don't throw that bait to Theo! I bet the proof will be insanely difficult and abstract.

Theo: Not at all. The proof is easy and very computational. I will be happy to show you—

Bob: Maybe not now. We will study the proof in Module 55.

Denny: But maybe Theo can tell us at least what $\mathbf{A}$ is?

Cindy: Yes, please tell us, Theo. If it's not too difficult. I'm curious.

Theo: Column number i of $\mathbf{A}$ is simply the function value $L(\vec{\mathbf{e}}_i)$, where $\vec{\mathbf{e}}_1, \ldots \vec{\mathbf{e}}_n$ are the standard basis vectors of $\mathbb{R}^n$.

For this matrix $\mathbf{A}$ we have $L = L_{\mathbf{A}}$.

What is this good for?

Frank: The argument really shows that at some level linear transformations are just another way of looking at matrix multiplication.

Alice: That is a very keen observation, Frank!

Frank: So why do we have to learn about them?

Alice: Thinking about linear transformations makes it easier to understand what all these other concepts we learned about mean, and how they are related to each other.

Frank: How?

Alice: We will discuss this in the second part of our conversation.

Take-home message

While in general there are various kinds of linear transformation, for each linear transformation L from the space of column vectors $\mathbb{R}^n$ into the space of column vectors $\mathbb{R}^m$ there exists an $m \times n$ matrix $\mathbf{A}$ such that $L(\vec{\mathbf{x}}) = L_{\mathbf{A}}(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$.

The columns of the matrix $\mathbf{A}$ for which $L = L_{\mathbf{A}}$ are the function values $L(\vec{\mathbf{e}}_i)$ of the standard basis vectors.