Conversation 31: Where do these names "eigenvectors" and "eigenvalues" come from?

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MATH3200: Applied Linear Algebra

Moving in one's own direction, Example 1

Denny: I am wondering where these strange names come from: "eigenvector" and "eigenvalue." They don't even sound like English words. Can you give us some insight into this, Theo?

Theo: Will be happy to. This is best understood if we consider the linear transformations $L_{\mathbf{A}}$.

Denny: Oh no! I don't like these linear transformations ...

Theo: Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$ and the linear transformation $L_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$.

Cindy: Was $L_{\mathbf{A}}$ the function such that $L_{\mathbf{A}}(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$?

Theo: Right.

Moving in one's own direction, Example 1 continued

Theo: Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$ and the linear transformation $L_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_{\mathbf{A}}(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$.

The function $L_{\mathbf{A}}$ is easy to understand: Every vector $\vec{\mathbf{x}} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ on the horizontal line gets stretched by a factor of 3 and mapped to a vector $\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 3x \\ 0 \end{bmatrix}$ on the same line, and every vector $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ y \end{bmatrix}$ on the vertical line gets compressed by a factor of 2 and mapped to a vector $\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 0.5y \end{bmatrix}$ on the same line. Note that the vectors on the axes, except for the origin, are the eigenvectors of $\mathbf{A}$.

So each eigenvector $\vec{\mathbf{x}}$ of $\mathbf{A}$ moves in its "own" direction, along the line $span(\vec{\mathbf{x}})$.

Moving in ones own direction, Example 2

Theo: Now consider $\mathbf{B} = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$

It is not immediately clear what the transformation $L_{\mathbf{B}} : \mathbb{R}^2 \to \mathbb{R}^2$ does.

But we already know that:

$$L_{\mathbf{B}}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \mathbf{B}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad L_{\mathbf{B}}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \mathbf{B}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus $L_{\mathbf{B}} : \mathbb{R}^2 \to \mathbb{R}^2$ corresponds to a fivefold stretch in the direction of the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and a twofold stretch in the direction of the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Again, each *eigenvector* $\vec{\mathbf{x}}$ moves along is *"own"* line *span*$(\vec{\mathbf{x}})$.

Question C31.1: What would be a good English word or short phrase for this phenomenon?

What happens along $span(\vec{\mathbf{x}})$?

Theo: Not easy to come up with one. So, mathematicians concatenated the German word "eigen," which means "one's own," with the English words "vector" and "value."

Denny: Awkward. But better than "own-value," which sounds like a mountain of debt!

Bob: Interesting.

Cindy: Yes, interesting.

But can you explain again what, exactly, the real eigenvalues mean in terms of the lines spanned by their eigenvectors?

The effects of eigenvalues on $L_{\mathbf{A}}$: Summary

Alice: Let $\vec{\mathbf{x}}$ be an eigenvector with eigenvalue λ of a square matrix $\mathbf{A}$.
Then the value of the linear transformation $L_{\mathbf{A}}(\vec{\mathbf{v}})$ for each vector $\vec{\mathbf{v}}$ on the line $span(\vec{\mathbf{x}})$ is a vector $\lambda\vec{\mathbf{v}}$ on the same line $span(\vec{\mathbf{x}})$.
More specifically:

- When $|\lambda| > 1$, then along the line $span(\vec{\mathbf{x}})$ the transformation $L_{\mathbf{A}}$ will be a stretch.
- When $|\lambda| = 1$, then $L_{\mathbf{A}}$ will preserve distances along the line $span(\vec{\mathbf{x}})$.
- When $0 < |\lambda| < 1$, then along the line $span(\vec{\mathbf{x}})$ the transformation $L_{\mathbf{A}}$ will be a compression.
- When $\lambda = 0$, then the line $span(\vec{\mathbf{x}})$ will be collapsed by the transformation $L_{\mathbf{A}}$ to the origin.
- When $\lambda < 0$, then along the line $span(\vec{\mathbf{x}})$ the transformation $L_{\mathbf{A}}$ will flip directions.

Bob: Let me see whether I got this straight. Assume that $\mathbf{A}$ has a full set of eigenvectors $\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$, respectively. Are you saying that in this case the linear transformation $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ geometrically looks essentially the same as the linear transformation $L_{\mathbf{D}} : \mathbb{R}^n \to \mathbb{R}^n$ of the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$$

only that it does these stretches and compressions in the directions of its eigenvectors $\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_n$ instead of the directions of the eigenvectors $\vec{\mathbf{e}}_1, \dots \vec{\mathbf{e}}_n$ of the diagonal matrix $\mathbf{D}$?

Question C31.2: Did Bob get this right?

Alice: Yes, this follows from what I said.

Can we use this observation to simplify calculations?

Frank: Could we then somehow set up things differently so that we can simplify all calculations by working with the diagonal matrix **D** instead of the matrix **A**?

Alice: Yes, this is possible when the matrix **A** is *diagonalizable*, which will be the case whenever it has a full set of eigenvectors.

Frank: How? I'm all for simplifying things.

Cindy: Can we talk about that some other time?

Alice: We will see soon how this works. But first we need to learn how to find eigenvalues and eigenvectors for a given matrix.