Conversation 32: Complex Eigenvalues

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MATH3200: Applied Linear Algebra

What about complex eigenvalues?

Frank: What's the deal with those "complex eigenvalues"? They don't make sense.

Denny: Are you saying, Frank, that they cannot be for real?

Frank: That's a good one! I mean: In this part of the course we always assume that all the vectors $\vec{\mathbf{x}}$ are in some $\mathbb{R}^n$, that their coordinates are real numbers. When you multiply such an $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ with a complex number like $\lambda = 1 - 2i$, then you get a vector that is not even in $\mathbb{R}^n$, and therefore cannot be equal to $\mathbf{A}\vec{\mathbf{x}}$. So, how can a complex number like $\lambda = 1 - 2i$ be an eigenvalue?

Theo: Formally, the eigenvalues of a matrix $\mathbf{A}$ are defined as the roots of the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$. Even when $\mathbf{A}$ has only real elements, some of these roots may not be real numbers. In particular, Frank's $\lambda = 1 - 2i$ may still be an eigenvalue. But Frank is right, there will be no real eigenvectors with such eigenvalues.

Frank: Thats' exactly my point: These λ are meaningless.

Rotations of the plane

Theo: Not at all! Complex eigenvalues of $\mathbf{A}$ give us important information about the geometry of the linear transformation $L_{\mathbf{A}}$.

Consider, for example, the matrix $\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

We have seen that the corresponding linear transformation $L_{\mathbf{R}_\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation of the plane by an angle α.

For this matrix we get $\mathbf{R}_\alpha - \lambda\mathbf{I} = \begin{bmatrix} \cos\alpha - \lambda & -\sin\alpha \\ \sin\alpha & \cos\alpha - \lambda \end{bmatrix}$ and

$\det(\mathbf{R}_\alpha - \lambda\mathbf{I}) = (\cos\alpha - \lambda)^2 + (\sin\alpha)^2$, which we can expand to
$\det(\mathbf{R}_\alpha - \lambda\mathbf{I}) = \lambda^2 - 2\lambda\cos\alpha + \cos^2\alpha + \sin^2\alpha$, which simplifies to
$\det(\mathbf{R}_\alpha - \lambda\mathbf{I}) = \lambda^2 - 2\lambda\cos\alpha + 1$.
The roots are $\cos\alpha + \sqrt{\cos^2\alpha - 1}$ and $\cos\alpha - \sqrt{\cos^2\alpha - 1}$.

Question C32.1: When are these roots real?

When are the eigenvalues of $\mathbf{R}_\alpha$ real?

Theo: When $|\cos\alpha| = 1$, then $\sqrt{\cos^2\alpha - 1} = 0$, and both $\cos\alpha + \sqrt{\cos^2\alpha - 1}$ and $\cos\alpha - \sqrt{\cos^2\alpha - 1}$ are real.

Question C32.2: For what angles α will we have $|\cos\alpha| = 1$?

Theo: This will happen when $\alpha = k\pi$ for some integer k.
Then $\cos\alpha$ is an eigenvalue of multiplicity 2, and

- either $\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix,
- or $\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

In the first case, every vector is an eigenvector with eigenvalue 1; in the second case, every vector is an eigenvector with eigenvalue -1.

In all other cases, the eigenvalues of $\mathbf{R}_\alpha$ form a pair of conjugate complex numbers.

It also works the other way around: Eigenplanes

Frank: All right, you gave us an example. But what is the "insight" that we supposedly get from complex eigenvalues?

Theo: It also works the other way around: Whenever a 2×2 matrix $\mathbf{A}$ with real elements does have a pair of conjugate complex eigenvalues, the linear transformation $L_{\mathbf{A}}$ involves a rotation of the plane. In the example that I showed you here, $L_{\mathbf{A}}$ was simply a rotation. But it could also be a composition of a rotation with stretches or compressions in one or two directions. I will be happy to show you why—

Bob: Maybe not now. We will explore an example in Module 67.

Cindy: And what if a matrix has a pair of conjugate complex eigenvalues and its order is $n \times n$ for some $n > 2$?

Theo: Then there exists a so-called *eigenplane*, that is a 2-dimensional subspace V of $\mathbb{R}^n$ so that $L_{\mathbf{A}}$ maps V onto V, and the restriction of $L_{\mathbf{A}}$ involves a rotation of this plane, perhaps composed with some deformations by stretches or compressions.

Take-home message

Question C32.3: Summarize in your own words what you learned from this conversation.

Complex roots of the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ are still considered eigenvalues.

When we are working with vectors in $\mathbb{R}^n$ then any eigenvalue that is not real does not have a corresponding eigenvector though.

Geometrically, a complex eigenvalue that is not real indicates that there is an *eigenplane* such the the restriction of the linear transformation $L_{\mathbf{A}}$ involves a rotation of this plane, perhaps composed with some stretches and/or compressions.