Conversation 34: Eigenvectors and Eigenvalues of Matrix Transposes and Inverse Matrices

Winfried Just
Department of Mathematics, Ohio University

MATH3200: Applied Linear Algebra

Cindy: I'm worried about these eigenvalues and eigenvectors of $\mathbf{A}^{-1}$ and of $\mathbf{A}^T$. For $\mathbf{A}^{-1}$, the eigenvectors are the same as for $\mathbf{A}$, but the eigenvalues are usually not, and for $\mathbf{A}^T$, the eigenvalues are the same as for $\mathbf{A}$, but the eigenvectors are usually not. How can I remember which is which, so as not to get mixed up on the final?

Theo: For the eigenvectors of $\mathbf{A}^{-1}$ this is easy to understand in terms of the linear transformations $L_{\mathbf{A}}$ and $L_{\mathbf{A}^{-1}}$.

Denny: Oh no! Not again linear transformations!

Theo: When $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ and when $\mathbf{A}$ is invertible, then the line $span(\vec{\mathbf{x}})$ will be mapped by $L_{\mathbf{A}}$ onto itself and every $\vec{\mathbf{y}}$ on the line $span(\vec{\mathbf{x}})$ will be stretched by a factor of λ.

Cindy: Could we take $\vec{\mathbf{y}} = \vec{\mathbf{x}}$ here?

Question C34.1: Would Theo's observation apply to $\vec{\mathbf{x}}$ itself?

Eigenvectors and eigenvalues of $\mathbf{A}^{-1}$, continued

Theo: Of course! Every vector $\vec{\mathbf{x}}$ is always in *span*$(\vec{\mathbf{x}})$.

The transformation $L_{\mathbf{A}^{-1}}$ is the inverse function of $L_{\mathbf{A}}$, and undoes the action of $L_{\mathbf{A}}$.

Question C34.2: How would one undo a stretch by a factor of λ?

Frank: By a corresponding compression.

Bob: Let me see whether I get this straight: Stretching a vector by a factor of λ amounts to multiplying the vector by a factor of λ, and what Frank called the "corresponding compression" amounts to multiplying this vector by a factor of $\frac{1}{\lambda}$, right?

Theo: Right.

Cindy: So that $L_{\mathbf{A}^{-1}}(\vec{\mathbf{x}}) = \mathbf{A}^{-1}\vec{\mathbf{x}} = \frac{1}{\lambda}\vec{\mathbf{x}}$ for our eigenvector $\vec{\mathbf{x}}$ of $\mathbf{A}$.

Theo: Exactly! Which implies that $\vec{\mathbf{x}}$ will be an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda}$.

Eigenvalues and eigenvectors of $\mathbf{A}^{-1}$: The theorem

Theo: This gives a proof of the following theorem:

Theorem

Let $\mathbf{A}$ *be an invertible matrix, and let* $\vec{\mathbf{x}}$ *be an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ.

Then $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}^{-1}$ *with eigenvalue* $\frac{1}{\lambda}$.

In other words, $\mathbf{A}$ and $\mathbf{A}^{-1}$ have the same eigenvectors, and the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

Denny: But what if $\lambda = 0$?

Question 37.3: How would you respond to Denny here?

Theo: The theorem assumes that $\mathbf{A}$ is invertible.

This implies that 0 is not an eigenvalue of $\mathbf{A}$, so that $\lambda \neq 0$.

Another proof of the theorem

Theorem

Let $\mathbf{A}$ be an invertible matrix, and let $\vec{\mathbf{x}}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ.

Then $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda}$.

Denny: Is there another proof without linear transformations?

Alice: Let $\vec{\mathbf{x}}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ.

Since $\mathbf{A}$ is invertible, 0 is not an eigenvalue of $\mathbf{A}$, so that $\lambda \neq 0$.

Then $\lambda\vec{\mathbf{x}} = \mathbf{A}\vec{\mathbf{x}}$, so that $\vec{\mathbf{x}} = \frac{1}{\lambda}\mathbf{A}\vec{\mathbf{x}} = \mathbf{A}\frac{1}{\lambda}\vec{\mathbf{x}}$, and

$\mathbf{A}^{-1}\vec{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{A}\frac{1}{\lambda}\vec{\mathbf{x}} = \mathbf{I}\frac{1}{\lambda}\vec{\mathbf{x}} = \frac{1}{\lambda}\vec{\mathbf{x}}$.

It follows that $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda}$. $\square$

Eigenvalues of the transpose $\mathbf{A}^T$

Cindy: So how about the eigenvalues and eigenvectors of the transpose $\mathbf{A}^T$?

Alice: The eigenvalues of $\mathbf{A}^T$ are the roots of the characteristic polynomial $\det(\mathbf{A}^T - \lambda\mathbf{I})$. In the formula $\mathbf{A}^T - \lambda\mathbf{I}$ we first take the transpose $\mathbf{A}^T$ of $\mathbf{A}$ and then subtract λ from each element of the diagonal.

Bob: Would we get the same result if we first subtract λ from each element of the diagonal of $\mathbf{A}$ and then take the transpose of the resulting matrix? I mean, is $\mathbf{A}^T - \lambda\mathbf{I} = (\mathbf{A} - \lambda\mathbf{I})^T$?

Alice: Yes. Since the diagonal does not change when you form a matrix transpose, this equality always holds.

Eigenvalues of the transpose $\mathbf{A}^T$, continued

Cindy: So, when $\mathbf{A}^T - \lambda\mathbf{I} = (\mathbf{A} - \lambda\mathbf{I})^T$, we also must have $\det(\mathbf{A}^T - \lambda\mathbf{I}) = \det((\mathbf{A} - \lambda\mathbf{I})^T)$, right?

Alice: Correct. Now we can use what we have learned about the determinant of the transpose.

Cindy: Like—that $\det(\mathbf{C}) = \det(\mathbf{C}^T)$ for any matrix $\mathbf{C}$?

Alice: Yes. This is true even when some elements of the matrix contain a variable, like λ in our $\mathbf{C} = \mathbf{A} - \lambda\mathbf{I}$.

Cindy: But this implies that $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic polynomial with the same roots, so that the eigenvalues of $\mathbf{A}^T$ must be the same as the eigenvalues of $\mathbf{A}$!

Denny: Cool! It is still weird though that $\mathbf{A}$ and $\mathbf{A}^T$ may not have the same eigenvectors, but the eigenvectors of $\mathbf{A}$ turn into left eigenvectors of $\mathbf{A}^T$ when you flip them around.

Eigenvalues and eigenvectors of matrix transposes $\mathbf{A}^T$

Theo: By "flipping around," do you mean taking transposes? As in the following theorem?

Theorem

Let $\mathbf{A}$ *be a square matrix, and let* $\vec{\mathbf{x}}$ *be an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ.

Then λ *is an eigenvalue of* $\mathbf{A}^T$,

but $\vec{\mathbf{x}}$ *is not always an eigenvector of* $\mathbf{A}^T$.

Moreover, $\vec{\mathbf{x}}^T$ *is a left eigenvector with eigenvalue* λ *of* $\mathbf{A}^T$,

which means that $\vec{\mathbf{x}}^T\mathbf{A}^T = \lambda\vec{\mathbf{x}}^T$.

Denny: Yeah, this is what I meant.

Turning a column vector $\vec{\mathbf{x}}$ into a row vector $\vec{\mathbf{x}}^T$.

Frank: Are these left eigenvectors good for anything?

Alice: We will talk about an important application of them another time.