Conversation 3: Adjacency Matrices of Digraphs

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MATH3200: Applied Linear Algebra

Review: A graph of friendships

Assume your instructor is nosy and wants to know about the friendships formed among the students in the class.

This information can be graphically represented by a *graph* G*:*

Draw a little circle for each student i.
These circles represent the *nodes* or *vertices* of G.

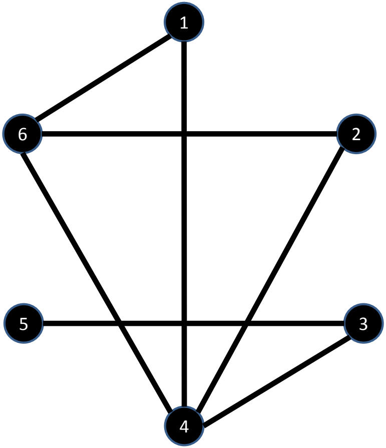
Then connect nodes i and j with a line segment if, and only if, i and j are friends.
These line segments represent the *edges* of G.

We are assuming here that i doesn't count as friend of i him- or herself, so that there are no loops in G from i to i.
This makes the graph G *loop-free.*

Also, we draw at most one line segment between any two given nodes. This makes the graph G *simple.*

We also assume that friendship is always reciprocated, so that when j is a friend of i, then also i is a friend of j.
This makes the graph G *undirected.*

Review: A graph of friendships



Review: The adjacency matrix of a graph

Consider the graph of friendships on the previous slide.
We can represent a graph G with n vertices in the form of its *adjacency matrix* $\mathbf{A} = [a_{ij}]_{n \times n}$,

where $a_{ij} = 1$ when there is an edge between i and j in G

and $a_{ij} = 0$ otherwise.

In our example, there are $n = 6$ nodes, so that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Review: Properties of adjacency matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} = [a_{ij}]_{m \times n}$$

- **A** is a *square matrix,* which means that $m = n$.
- All elements a_{ii} on the *(main) diagonal,* aka *diagonal elements,* are zero.
- **A** is *symmetric,* which means that $a_{ij} = a_{ji}$ for all i, j.

We can see here that the adjacency matrix for our particular graph has these properties. But one can show that any adjacency matrix of any undirected loop-free graph has the three properties listed above.

Directed graphs aka digraphs

Bob: It says on the last slide "undirected graphs." Makes me wondering: Are there also directed ones?

Theo: Yes there are. Pick nodes $i = 1, \ldots, n$ for some n.

Cindy: Like in undirected graphs?

Theo: So far, yes. But then connect nodes i and j with arrows instead of line segments.
These arrows are called *arcs* of G.

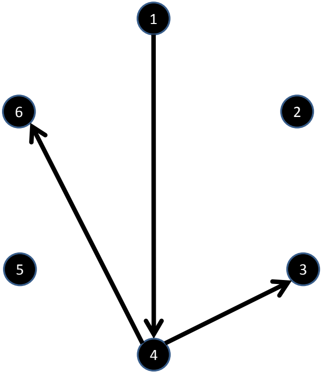
Cindy: So that an arrow, or arc as you say, could point from i to j or from j to i?

Theo: Exactly. If the arc goes from i to j, then
i is called the *source* of the arc,
and j is called the *target* of the arc.

Bob: Can you show us an example?

An example of a digraph

Theo: In this example, node 4 is the target of one arc with source 1 and the source of two arcs, with targets 3 and 6, respectively.



What are digraphs good for?

Frank: But these digraphs don't make sense.

Theo: Why not?

Frank: Because if i is friends with j, then j is also friends with i, so the direction of the arrow would be meaningless.

Question C3.1: Would you agree or disagree with Frank?
Why or why not?

Denny: Friendships are not always reciprocated.
I could tell you stories ...

Cindy: Maybe not now. But perhaps the arcs in digraphs could represent something other than friendships?

Other applications of digraphs

Bob: Like, for example, links between web pages that are represented by nodes. An arc from page i to page j would then indicate that there is a link at page i to page j.

Frank: What if there are multiple links to page j at the same page i?

Denny: Could we then draw multiple arrows from i to j, one for each such link?

Theo: Yes we could. If there are multiple arcs between some pair of nodes, the resulting structure D would still be a *directed graph* aka *digraph,* but it would no longer be a simple digraph.

Denny: Cool! Are there other applications of digraphs?

Alice: There are many. For example, consider the spread of an infectious disease.

Frank: What does this have to do with digraphs?

The spread of an infection

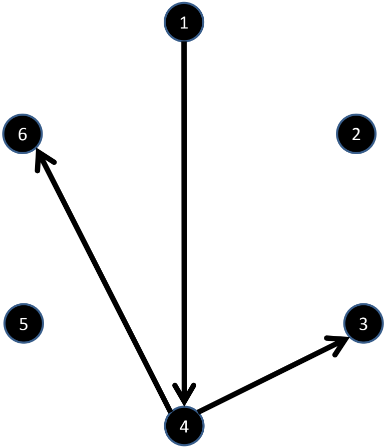
Alice: Consider an infectious disease, like the flu, that spreads among a group of people that is called a *population* in this context. Here nodes would represent people, as in graphs of friendships. An arc from node i to node j would signify that i infects j.

Within a population of six students, the infection might spread as follows:

1. Student 1, the so-called *index case,* gets infected from somebody outside this group.
2. Student 1 infects student 4.
3. Student 4 infects students 3 and 6.
4. No further infections occur within this group.

What do we get if we represent this scenario as a directed graph?

The digraph of Theo's example!



The adjacency matrix of a digraph

Cindy: What does all this have to do with matrices?

Theo: We can also represent a digraph D with n nodes by an adjacency matrix $\mathbf{A} = [a_{ij}]_{n \times n}$. Here a_{ij} would be the *number of arcs from node i to node j*.

Cindy: So that in the example we have just seen, we would have $a_{34} = 1$ and $a_{43} = 0$?

Question C3.2: Is Cindy right?

Alice: You have a point, Cindy. We may now have $a_{ij} \neq a_{ji}$.
But it is actually the other way around:
In this example the arc points from node 4 to node 3, so that $a_{43} = 1$ and $a_{34} = 0$.

Cindy: Oops! Sorry! I had mixed up the order of the subscripts.

The adjacency matrix for our example

Alice: Here is the complete adjacency matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [a_{ij}]_{m \times n}$$

It still is a square matrix with zeros on the main diagonal, but it is no longer symmetric.

More about adjacency matrices

Frank: But wait! Why don't we just define the entry a_{ij} of the adjacency matrix as 0 if there is no arc from i to j and as 1 otherwise? This would give us the same adjacency matrix as on the previous slide.

Denny: But perhaps it does not always give the same result?

Cindy: Could we also define a_{ij} as the number of edges between i and j in the adjacency matrix of an undirected graph?

Question C3.3: How would you reply to Frank, Denny, and Cindy?

Theo: The two definitions give us the same adjacency matrices for all simple directed or undirected graphs, that is, when the number of arcs or edges between any given two nodes is always either 0 or 1. In particular, both definitions give us the same adjacency matrix for the simple digraph in example.

However, when a graph has multiple edges or arcs between the same pair of nodes, we need to define the adjacency matrix in terms of their number.

Take-home message

- *Directed graphs* aka *digraphs* are structures similar to graphs.
- While in (undirected) graphs nodes are connected by edges, in digraphs nodes are connected by *arcs*.
- An arc can be depicted by a little arrow that points from node i (its *source*) to node j (its *target*).
- Digraphs have many applications. For example, they allow us to study the transmission of infectious diseases and the linkage structure of the WWW.
- The element a_{ij} of the *adjacency matrix* $\mathbf{A}$ of a digraph is the *number of arcs from node i to node j*.
- Adjacency matrices of digraphs are usually not symmetric.