Conversation 5: More about proofs. Translating assumptions of theorems

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MATH 3200: Applied Linear Algebra

A question from Module 8

Bob: It says here in Module 8:

"Let $\mathbf{A} = [a_{ij}]_{2\times 2}$ and $\mathbf{B} = [b_{ij}]_{2\times 2}$ be two symmetric 2×2 matrices of numbers such that $a_{11} = a_{22}$ and $b_{11} = b_{22}$. Prove that $\mathbf{AB} = \mathbf{BA}$."

Let's give this a try!

Cindy: But I never know how to get started with these proofs!

Alice: (gently) OK. How *might* the start of a proof look like?

Cindy: (sigh) Would I write: "Let $\mathbf{A} = [a_{ij}]_{2\times 2}$ and $\mathbf{B} = [b_{ij}]_{2\times 2}$ be two symmetric 2×2 matrices of numbers"?

Alice: You could.

Cindy: But I don't like this "$[a_{ij}]_{2\times 2}$" notation.

Bob: Then use a different one that shows all elements of both matrices.

Choosing a convenient notation

Cindy: Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Like this, Bob?

Bob: Good!

Cindy: But I don't like all these subscripts. Could I write:

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

Alice: This notation might also be fine; let's give it a try.
Now what do you want to prove, Cindy?

Cindy: That $\mathbf{AB} = \mathbf{BA}$.

Alice: So what do you need to do next?

Does this work?

Cindy: (sigh) I need to calculate with symbols

$$\mathbf{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \text{ and } \mathbf{BA} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and verify that both calculations give the same result.

Bob: High 5, Cindy! This would be the thing to do here.

Frank: This isn't going to work out. Forget it!

Bob: Don't be so negative, Frank! It will work out fine.

Question C5.1: Who is right here? Frank or Bob?

Cindy: Please don't start an argument.

Let me do the calculations and we will see who's right.

Why doesn't this work?

Cindy: I get:

$$\mathbf{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + ch \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & eb + fd \\ ga + fg & hc + hd \end{bmatrix}$$

But I can't see why the two products are equal.

Frank: Of course not!

Bob: Stop it, Frank!!

Frank: It's not your fault, Cindy. As it says in Module 8, matrix multiplication is not commutative, so $\mathbf{AB} \neq \mathbf{BA}$.

Denny: Wow! The prof asked us to prove a wrong theorem. Some prof—

Translating the assumption of symmetry

Theo: Not so fast, Denny! **Sometimes $\mathbf{AB} = \mathbf{BA}$**, and the theorem says that this will be the case under the assumption that the matrices are symmetric, and that $a_{11} = a_{22}$ and $b_{11} = b_{22}$.

Cindy didn't make this assumption when she set up her notation.

Cindy: So I should have used the one with all those subscripts?

Alice: Not necessarily.

We can translate the assumptions into your notation.

Question C5.2: What does "$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is symmetric" mean?

Cindy: That $b = c$.

And $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ is symmetric when $f = g$.

Translating the assumption of symmetry, continued

Alice: So how could you write in symbols: "Let $\mathbf{A}, \mathbf{B}$ be two symmetric 2×2 matrices"?

Cindy: Let $\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ f & h \end{bmatrix}$

In each matrix, I can use the same letter for the off-diagonal elements, since they must be equal.

Now what do I do next?

Alice: You can figure this out, Cindy.

Think about the theorem that you want to prove.

Question C5.3: Should Cindy now calculate the products **AB** and **BA** of these matrices and show that they are equal?

Translating the other two assumptions

Theo: Not yet. We also need to take into account the other two assumptions that $a_{11} = a_{22}$ and that $b_{11} = b_{22}$.

Cindy: But wouldn't I need to use those subscripts for them?

Alice: You can do it without subscripts.

Question C5.4: What are a_{11} and a_{22} for $\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$?

Cindy: $a_{11} = a$ and $a_{22} = d$.

And $b_{11} = e$, $b_{22} = h$ for $\mathbf{B} = \begin{bmatrix} e & f \\ f & h \end{bmatrix}$

Alice: So when $a_{11} = a_{22}$ and $b_{11} = b_{22}$ —

Cindy: — then $\mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ f & e \end{bmatrix}$

Wow!! I can make do with a lot fewer different letters!

Theo's 2 cents

Theo: The most professional way to start your proof would have been this:

Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

be two 2×2 matrices that satisfy the assumptions $a_{11} = a_{22}$ and $b_{11} = b_{22}$. Then $a_{12} = a_{21}$ and $b_{12} = b_{21}$ by symmetry, so that we can simplify our notation and represent each matrix using only two letters.

So let us assume $\mathbf{A} = \begin{bmatrix} a & c \\ c & a \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b & d \\ d & b \end{bmatrix}$

Cindy: Now let me calculate **AB** and **BA** and compare the results.

Cindy completes the proof

$$\mathbf{AB} = \begin{bmatrix} a & c \\ c & a \end{bmatrix} \begin{bmatrix} b & d \\ d & b \end{bmatrix} = \begin{bmatrix} ab + cd & ad + cb \\ cb + ad & cd + ab \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} b & d \\ d & b \end{bmatrix} \begin{bmatrix} a & c \\ c & a \end{bmatrix} = \begin{bmatrix} ba + dc & bc + da \\ da + bc & dc + ba \end{bmatrix}$$

Denny: Same thing! End of the proof. As they write in math: $\square$

Frank: High 5, Cindy!

Bob: But wait a minute! It says here in the assumptions that **A** and **B** are matrices whose elements are **numbers.** Why is that?

Question C5.5: Where did we use this additional assumption?

Alice: To see that the results of the two calculations are really the same, we need that $ab = ba$, *etc.* This will be the case when the elements are numbers, but not necessarily when these elements are, for example, themselves matrices.

Take-home message

- At the beginning of a proof, you need to express the assumptions of a theorem in terms of your symbolic notation.
- When some of the assumptions imply that certain quantities are equal, you may be able to use fewer symbols than you would need otherwise.
- The best strategy is to defer any symbolic calculations until after you have translated the assumptions.
- If your calculations don't work out as expected, double-check whether you have correctly translated the assumptions of the theorem.
- Make sure you have used every assumption of the theorem somewhere in your proof.