Conversation 6: More about proofs. Translating conclusions of theorems

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MATH 3200: Applied Linear Algebra

Another question from Module 8

Bob: Shall we do the next question of the module?

"Let **A** and **B** be two symmetric 2×2 matrices of numbers. Prove that **AB** and **BA** are transposes of each other."

Cindy: Wouldn't we just do the same thing here as in the previous proof?

The Others: Go for it, Cindy!

Cindy: I mean, start like Theo suggested for the previous one:

Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

be two symmetric 2×2 matrices of numbers. Then $a_{12} = a_{21}$ and $b_{12} = b_{21}$ by symmetry, so that we can simplify our notation and represent each matrix using only three letters. So we let:

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$$

Cindy continues her proof

Denny: Wow!!

Alice: You got the hang of it, Cindy!

Frank: Same thing indeed! Repetitive. They shouldn't make us learn how to do proofs in a class for engineering students.

Bob: Stop it, Frank! Let Cindy do her work.

Cindy: Now let us calculate **AB** and **BA** and compare the results:

$$\mathbf{AB} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ e & f \end{bmatrix} = \begin{bmatrix} ad + be & ae + bf \\ bd + ce & be + cf \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} d & e \\ e & f \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} da + eb & db + ec \\ ea + fb & eb + fc \end{bmatrix}$$

But I don't see why the results would be the same.

Frank: They aren't. Look at the lower left corners.

Denny: Again we were assigned a wrong proof! This prof really—

Where did Cindy go wrong?

Cindy: Did I mess up again?

Question C6.1: What is wrong with Cindy's proof?

Alice: Nothing so far.

Cindy: How could the calculations then give different results?

Alice: Were they *supposed to* give the same result?

Bob: Let's read the question again:

"Let **A** and **B** be two symmetric 2×2 matrices of numbers.
Prove that **AB** and **BA** are transposes of each other."

Cindy: So I don't need to show that the two products are equal?

Theo: You need to translate not only the assumptions of a theorem into formulas, but also its *conclusion.*
Which is in our example: "are transposes of each other."

Bob: This can be written as: $(\mathbf{AB})^T = \mathbf{BA}$.

Cindy and Bob complete the proof

Cindy: So I need to compute

$$(\mathbf{AB})^T = \begin{bmatrix} ad+be & ae+bf \\ bd+ce & be+cf \end{bmatrix}^T = \begin{bmatrix} ad+be & bd+ce \\ ae+bf & be+cf \end{bmatrix}$$

and then compare this with what I got earlier for **BA**:

$$\mathbf{BA} = \begin{bmatrix} da+eb & db+ec \\ ea+fb & eb+fc \end{bmatrix}$$

Bob: Since we have assumed that the elements of **A** and **B** are numbers, and since multiplication of numbers is commutative, we can see that results of Cindy's calculations contain the exact same elements in the corresponding positions.
We conclude that $(\mathbf{AB})^T = \mathbf{BA}$. □

Cindy: Thank you, Bob, for saying this so clearly!

The others: Great job, Cindy and Bob!

Take-home message

- We had seen in the previous conversation that at the beginning of a proof, you need to express the assumptions of a theorem in terms of your symbolic notation. When some of the assumptions imply that certain quantities are equal, you may be able to use fewer symbols than you would need otherwise.
- In this conversation we saw that you also need to translate the conclusion of the theorem into symbols. Here "conclusion" refers the property that the theorem asserts and that you want to prove.
- The best strategy is to defer any symbolic calculations until after you have translated both the assumptions and the conclusion.
- If your calculations don't work out as expected, double-check whether you have correctly translated the assumptions and the conclusion of the theorem.