Lecture 11: Introduction to Systems of Linear Equations

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MATH3200: Applied Linear Algebra

Familiar equations of lines in the x-y-plane $\mathbb{R}^2$

In this chapter, the symbol $\mathbb{R}^2$ will always denote the set of all 2-dimensional column vectors of real numbers that represent points in the x-y-plane; the symbol $\mathbb{R}^3$ will always denote the set of all 3-dimensional column vectors of real numbers; and so on.

Consider the familiar equation $y = ax + b$.

This equation defines a line in the x-y-plane that consists of all

vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ whose coordinates satisfy this equation.

Question L11.1: Can every line in the x-y-plane be defined in this way?

No. For example, the y-axis $x = 0$ cannot be defined by an equation $y = ax + b$.

Only lines that are not vertical can be defined in this way.

The general form of equations of lines in the x-y-plane $\mathbb{R}^2$

Now consider equations of the form: $a_1x + a_2y = b$.

- For $a_1 := -a$ and $a_2 := 1$ we get $-ax + y = b$.

  This gives the same lines $y = ax + b$ as on the previous slide.

- For $a_1 := 1$ and $a_2 := 0$ we get $x = b$.

  This gives us all vertical lines for suitable choices of b.

Thus we can see that the *most general form of a linear equation* in two variables x, y is:

$$a_1x + a_2y = b,$$

where a_1, a_2, b are constants.

Equations of planes in the three-dimensional space $\mathbb{R}^3$

Similarly, consider an equation of the form

$$a_1x + a_2y + a_3z = b,$$

where a_1, a_2, a_3, b are constants.

This equation defines a plane in the x-y-z-space $\mathbb{R}^3$ that consists

of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ whose coordinates satisfy this equation.

For example, if $a_1 = a_2 = 0$, while $a_3 = 4$, and $b = 12$, then
$a_1x + a_2y + a_3z = b$ becomes $4z = 12$ and
defines a horizontal plane that intersects the z-axis at 3.

Equations of hyperplanes in $\mathbb{R}^n$

Notice that the common feature of the linear equations

$a_1 x + a_2 y = b$ and $a_1 x + a_2 y + a_3 z = b$

is that the set of their solutions is a subspace of the relevant space (the x-y-plane $\mathbb{R}^2$ or the x-y-z-space $\mathbb{R}^3$) of one dimension less.

More generally, consider an equation of the form

$$a_1 x_1 + a_2 x_2 + \dots a_n x_n = b,$$

where $a_1, a_2, \dots a_n, b$ are constants, and $x_1, x_2, \dots, x_n$ are variables.

The solutions of this equation form what is called a *hyperplane* in the space $\mathbb{R}^n$ of all n-dimensional column vectors.

Hyperplanes are subspaces of $\mathbb{R}^n$ of dimension $n - 1$.

Systems of linear equations

A *system of m linear equations in n variables* is an expression

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We assume here that all a_{ij} and b_i are given scalar constants.

A column vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ of numbers such that

all equations are satisfied is *a solution* of the system.

Example 1: Two lines with different slopes

Consider the following system of $m = 2$ linear equations with $n = 2$ variables:

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ 2x_1 + 2x_2 &= 6 \end{aligned}$$

The first equation defines a line with slope -2 in the x_1-x_2-plane. The second equation defines a line with slope -1 in the x_1-x_2-plane.

The two lines intersect in a single point.

The column vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the unique a solution of the above system.

Uniqueness of solutions is *typical,* but *by no means universal* when m, the number of equations, is equal to n, the number of variables.

Example 2: Two parallel lines

Consider the following system of $m = 2$ linear equations with $n = 2$ variables:

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Both equations define lines with slope -2 in the x_1-x_2-plane.

Question L11.2: Where do these lines intersect the x_2-axis?

The x_2-axis consists of all points with $x_1 = 0$. By setting $x_1 := 0$ in the above equations, we find that the first line passes through the point $[0, 4]$, while the second line passes through the point $[0, 3]$.

These lines are parallel, but distinct.

This system has no solutions whatsoever.

It is *inconsistent* aka *overconstrained* aka *overdetermined.*

Examples 3: Two equations for the same line

Consider the following system of $m = 2$ linear equations with $n = 2$ variables:

$$2x_1 + x_2 = 4$$
$$4x_1 + 2x_2 = 8$$

Both equations define the same line with slope -2 that passes through the point $[0, 4]$ of the x_1-x_2-plane.

This system is consistent, but it has more than one solution.

More precisely, each column vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ 4 - 2x_1 \end{bmatrix}$ is a solution.

This system is *underconstrained* aka *underdetermined,* which means that it has **infinitely many solutions.**

Could a linear system have exactly 2 solutions?

Question L11.3: Could there be a system of 2 equations with 2 variables that has exactly 2 solutions?

Not a system of *linear* equations. Since each linear equation with two variables defines a line in $\mathbb{R}^2$, if these lines intersect in 2 points, they must be the same line, so that the system will have infinitely many solutions.

More generally, *any* system of linear equations will always have either exactly one solution, infinitely many solutions, or no solutions at all.

Example 4: What if all coefficients a_{ij} are zero?

Consider an equation of the form

$$0x_1 + 0x_2 = b.$$

This is a linear equation with $a_1 = a_2 = 0$.

- When $b \neq 0$, this equation is *inconsistent* and does not have any solution.

- When $b = 0$, every vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a solution,

  and this equation is *underdetermined.*

Linear equations as the one above, with all coefficients a_{ij} equal to zero, are certainly not particularly interesting. But we will see soon why we need to consider them in this course.

Example 5: A system of 2 equations in 3 variables

Consider the following system of $m = 2$ linear equations with $n = 3$ variables:

$$2x_1 + x_2 - x_3 = 0$$
$$3x_1 + x_2 - x_3 = 0$$

This system is underdetermined.

Every column vector $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix}$ is a solution.

The above system is also an example of a *homogenous system,*

which means that $b_1 = b_2 = \cdots = b_m = 0$.

For homogeneous systems the vector $\vec{\mathbf{0}}$ with $x_1 = x_2 = \cdots = x_n = 0$ is always a solution.

Example 6: Another system of 2 equations in 3 variables

Consider the following system of $m = 2$ linear equations with $n = 3$ variables:

$$2x_1 + x_2 - x_3 = 0$$
$$4x_1 + 2x_2 - 2x_3 = 1$$

Question L11.4: Is this system homogeneous? What can you say about its solution set?

This system is *not* homogeneous, because only $b_1 = 0$, but $b_2 \neq 0$.

The system is *inconsistent,* which can be seen by dividing both sides of the second equation by 2.

In general, when the number n of variables of a system of linear equations exceeds the number m of its equations, then the system must be either inconsistent or underconstrained.

Take-home message: Linear equations

The general form of a linear equation in n variables is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b,$$

where $a_{11}, \ldots, a_{1n}, b$ are fixed scalars and $x_1, \ldots, x_n$ are variables.

For $n = 2$ variables, this equation defines a line in $\mathbb{R}^2$.

For $n = 3$ variables, this equation defines a plane in $\mathbb{R}^3$.

For n variables, this equation defines a *hyperplane* in $\mathbb{R}^n$.

In this lecture we have always treated $\mathbb{R}^n$ as a set of column vectors. For reasons that will soon become apparent, we will do so throughout this chapter and whenever we consider solutions of systems of linear equations.

Take-home message: Systems of linear equations

A *system of m linear equations in n variables* is an expression

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We assume that all a_{ij} and b_i are given scalar constants.

A column vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ of numbers such that

all equations are satisfied is *a solution* of the system.

When $b_1 = b_2 = \cdots = b_m = 0$, the system is *homogeneous.*

Take-home message: Properties of the solution set

Consider a system of m linear equations in n variables.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The system is *consistent* if it has at least one solution.

Homogeneous systems are always consistent.

Consistent systems may have either one or infinitely many solutions. In the latter case the system is *underdetermined* aka *underconstrained.*

The system is *inconsistent* aka *overdetermined* aka *overconstrained* if it has no solution.

When the number n of variables exceeds the number m of equations, the system will be either underdetermined or inconsistent.