Lecture 12: Matrix Representations of Systems of Linear Equations

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MATH3200: Applied Linear Algebra

The coefficient matrix of a system of linear equations

Consider a system of m linear equations in n variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The *coefficient matrix* of this system is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Example of a coefficient matrix

Consider a system of 3 linear equations in 4 variables.

$$
\begin{aligned}
3x_1 + 4x_2 - 6x_3 &= 11 \\
x_1 - x_3 + 5x_4 &= 7 \\
x_2 + 2x_3 + 8x_4 &= 0
\end{aligned}
$$

The coefficient matrix of this system has elements $a_{11} = 3$ and $a_{12} = 4$:

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 3 & 4 & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}
$$

Question L12.1: What is a_{13}?

Example of a coefficient matrix

Consider a system of 3 linear equations in 4 variables.

$$\begin{aligned} 3x_1 + 4x_2 - 6x_3 &= 11 \\ x_1 - x_3 + 5x_4 &= 7 \\ x_2 + 2x_3 + 8x_4 &= 0 \end{aligned}$$

The coefficient matrix of this system has elements $a_{11} = 3, a_{12} = 4$, and $a_{13} = -6$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -6 & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Question L12.2: What is a_{14}?

Example of a coefficient matrix

Consider a system of 3 linear equations in 4 variables.

$$
\begin{aligned}
3x_1 + 4x_2 - 6x_3 &= 11 \\
x_1 - x_3 + 5x_4 &= 7 \\
x_2 + 2x_3 + 8x_4 &= 0
\end{aligned}
$$

The coefficient matrix of this system has elements
$a_{11} = 3, a_{12} = 4, a_{13} = -6$, and $a_{14} = 0$:

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -6 & 0 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}
$$

Question L12.3: What is a_{21}?

Example of a coefficient matrix

Consider a system of 3 linear equations in 4 variables.

$$
\begin{aligned}
3x_1 + 4x_2 - 6x_3 &= 11 \\
x_1 - x_3 + 5x_4 &= 7 \\
x_2 + 2x_3 + 8x_4 &= 0
\end{aligned}
$$

The coefficient matrix of this system has elements

$a_{11} = 3, a_{12} = 4, a_{13} = -6, a_{14} = 0$, and $a_{21} = 1$:

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -6 & 0 \\ 1 & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}
$$

Question L12.4: What is a_{22}?

Example of a coefficient matrix

Consider a system of 3 linear equations in 4 variables.

$$\begin{aligned} 3x_1 + 4x_2 - 6x_3 &= 11 \\ x_1 - x_3 + 5x_4 &= 7 \\ x_2 + 2x_3 + 8x_4 &= 0 \end{aligned}$$

Here $a_{22} = 0$.

The entire coefficient matrix of this system is:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -6 & 0 \\ 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & 8 \end{bmatrix}$$

Question L12.5: Is there only one system of linear equations with this coefficient matrix?

No.

Infinitely many systems with the same coefficient matrix

For example, the systems

$$\begin{aligned} 3x_1 + 4x_2 - 6x_3 &= 11 \\ x_1 - x_3 + 5x_4 &= 7 \\ x_2 + 2x_3 + 8x_4 &= 0 \end{aligned}$$

and

$$\begin{aligned} 3x_1 + 4x_2 - 6x_3 &= \pi \\ x_1 - x_3 + 5x_4 &= 7.43 \\ x_2 + 2x_3 + 8x_4 &= -40.8 \end{aligned}$$

have the same coefficient matrix. In fact, the numbers

$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ on the right-hand side could form any vector in $\mathbb{R}^3$.

There are infinitely many systems with the same coefficient matrix.

The extended matrix of a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

The *coefficient matrix* **A** does *not* give us complete information about this system.

But the *extended* aka *augmented* matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ *does:*

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

Example of an extended matrix

Consider a system of 3 linear equations in 4 variables.

$$
\begin{aligned}
3x_1 + 4x_2 - 6x_3 &= 11 \\
x_1 - x_3 + 5x_4 &= 7 \\
x_2 + 2x_3 + 8x_4 &= 0
\end{aligned}
$$

The extended matrix of this system is:

$$
[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -6 & 0 & 11 \\ 1 & 0 & -1 & 5 & 7 \\ 0 & 1 & 2 & 8 & 0 \end{bmatrix}
$$

Question L12.6: Is there only one system of linear equations with this extended matrix?

Yes.
The extended matrix gives a complete description of the system.

The left-hand side of a system as a matrix product

Let **A** be the coefficient matrix of a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then for $\vec{\mathbf{x}} = [x_1, x_2, \dots, x_n]^T$ we have

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ ? \\ \vdots \\ ? \end{bmatrix}$$

The left-hand side of a system as a matrix product

Let **A** be the coefficient matrix of a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then for $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]^T$ we have

$$\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ ? \end{bmatrix}$$

The left-hand side of a system as a matrix product

Let $\mathbf{A}$ be the coefficient matrix of a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then for $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]^T$ we have

$$\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

The matrix form of a system of linear equations

Let **A** be the coefficient matrix of a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then the system can be expressed in *matrix form* as follows:

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \quad \text{where} \quad \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Example of a system in matrix form

Consider a system of 3 linear equations in 4 variables.

$$\begin{aligned} 3x_1 + 4x_2 - 6x_3 &= 11 \\ x_1 - x_3 + 5x_4 &= 7 \\ x_2 + 2x_3 + 8x_4 &= 0 \end{aligned}$$

This system can be expressed in matrix form as:

$$\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 3 & 4 & -6 & 0 \\ 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{\mathbf{b}}$$

Summary

Consider a system of m linear equations in n variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}$$

are the *coefficient matrix* and the *extended* or *augmented* matrix of the system, respectively.

The system can be written in *matrix form* as $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$,

where $\quad \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad$ and $\quad \vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$