Lecture 13: Matrices in (Row) Echelon Form and More on Back-Substitution

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MATH3200: Applied Linear Algebra

Review: A system from Conversation 12

Consider the following system of linear equations:

$$\begin{aligned} x_1 \qquad\qquad &= 3 \\ x_2 \qquad &= 2 \\ 0 &= 1 \end{aligned}$$

We could immediately see that the system is inconsistent.

The extended matrix of this system is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: A second system from Conversation 12

Consider the following system of linear equations:

$$\begin{aligned} x_1 \qquad\qquad &= 3 \\ x_2 \qquad &= 2 \\ 0 &= 0 \end{aligned}$$

We could immediately see that the system is underdetermined

and that its solution set consists of all vectors $\begin{bmatrix} 3 \\ 2 \\ x_3 \end{bmatrix}$

The extended matrix of this system is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Review: A third system from Conversation 12

Consider the following system of linear equations:

$$\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -3 \\ & & x_2 & = & -2 \end{array}$$

We could solve the system by back-substitution.

The unique solution is the vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$

The extended matrix of this system is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

Review: A fourth system from Conversation 12

Consider the following system of linear equations:

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 1 \\ & & & & x_3 & = & 2 \end{array}$$

We could solve the system by back-substitution.

The unique solution is the vector $\begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$

The extended matrix of this system is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Review: A fifth system from Conversation 12

Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & 3x_3 & - & 2x_4 & = & 0 \\ & & & & x_3 & - & x_4 & = & 1 \\ & & & & & & x_4 & = & 2 \end{array}$$

We could solve the system by back-substitution.

The solution set consists of all vectors $\begin{bmatrix} x_2 - 5 \\ x_2 \\ 3 \\ 2 \end{bmatrix}$

The extended matrix of this system is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

What do all of these examples have in common?

We have seen that the systems with the following augmented matrices could all be solved relatively easily by back-substitution:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Question L13.1: Try to describe in your own words the features that all these matrices have in common.

They are all in *row echelon form* or simply *echelon form.*
This form will be rigorously defined on the next slide.

The definition of matrices in echelon form

A matrix is in *row echelon form* or simply *echelon form,* if:

(R1) All *zero rows,* that is, rows with only zeros, appear below all *nonzero rows* when both types are present.

(R2) The first nonzero entry in any nonzero row is 1.

(R3) All elements in the same column below the first nonzero element of a nonzero row are 0.

(R4) The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.

Note on the terminology: Some textbooks do not require condition (R2) in this definition. We will sometimes say that a matrix that satisfies conditions (R1), (R3), and (R4) is in *generalized echelon form.*

The first nonzero element of a nonzero row of a matrix is often called the *leading element* of that row.

More examples

Question L13.2: Which of the following matrices are not in echelon form?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & 17 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -3 & 0 & 17 \\ 0 & 0 & 1 & -1 & -13 \\ 0 & 0 & 0 & 1 & 33 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 4 & -6 & 0 & 13 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 1 & 6 & 1 & 0 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & 1.5 \\ 0 & 1 & 0.25 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} 1 & 2 & 3 & 1.5 \\ 0 & 1 & -1 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix **C** is not in echelon form, because it has a zero row that precedes a nonzero row. Matrix **D** is not in echelon form either because the leading 1 in the second row is further to the right than the leading 1 in the last row.

Matrices in reduced echelon form

A matrix is in *reduced row echelon form* or simply *reduced echelon form,* if:

(R1) All *zero rows,* that is, rows with only zeros, appear below all *nonzero rows* when both types are present.

(R2) The first nonzero entry in any nonzero row is 1.

(R3+) All elements in the same column *as* the first nonzero element of a nonzero row are 0.

(R4) The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.

Note that the only difference between the definitions of echelon form and reduced echelon form of a matrix is that we replaced the word "below" in the definition of condition (R3) with the word "as" in condition (R3+).

More examples

The matrix $\mathbf{A} := \begin{bmatrix} 2 & 2 & -3 & 0 & 17 \\ 0 & 0 & 1 & -1 & -13 \\ 0 & 0 & 0 & 3 & 33 \end{bmatrix}$ is in

generalized echelon form, but not in echelon form,
as the first nonzero elements of rows 1 and 3 are not equal to 1.

The matrix $\mathbf{B} := \begin{bmatrix} 1 & 2 & -3 & 0 & 17 \\ 0 & 0 & 1 & -1 & -13 \\ 0 & 0 & 0 & 1 & 33 \end{bmatrix}$ is in echelon form,

but not in reduced echelon form, as there are nonzero elements in the same columns as the leading 1s in rows 2 and 3.

The matrix $\mathbf{C} := \begin{bmatrix} 1 & 2 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -13 \\ 0 & 0 & 0 & 1 & 33 \end{bmatrix}$ is in

reduced echelon form. Note that the nonzero entries other than the leading 1s don't destroy these property, because there are no leading ones in the same columns.

The row echelon form and back-substitution

Any system of linear equations that is represented by an extended matrix in row echelon form can be solved by back-substitution, as illustrated in Conversation 12.

This method is straightforward when the solution is unique.

However, when the system is underdetermined, you need to represent the solution set by judiciously choosing one or more *free-variables* or *free parameters.* You need to choose these free variables in such a way that **every choice of values for these free variables will give you a solution of the system, and all solutions can be obtained in this way.**

Consider the extended matrix $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Question: What are the legitimate choices of free variables for representing the solution set of the linear system with variables x_1, x_2, x_3, x_4 and extended matrix $[\mathbf{A}, \vec{\mathbf{b}}]$?

The choice of free variables: An example

The system with extended matrix $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ is

$$\begin{array}{ccccc} x_1 & + & 0.5x_2 & & & = & 3 \\ & & & & x_3 & = & 0 \end{array}$$

None of the equations contains any information on x_4.

Question L13.3: So, what are our options for making x_4 a free variable?

Therefore, x_4 must be one of our free variables.

The second equation implies that $x_3 = 0$.

Question L13.4: So, what are our options for making x_3 a free variable?

The variable x_3 *cannot* be free.

The choice of free variables: An example

The system with extended matrix $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ is

$$\begin{array}{rcrcrcl} x_1 & + & 0.5x_2 & & & = & 3 \\ & & & & x_3 & = & 0 \end{array}$$

Question L13.5: What are our options for making x_1 or x_2 a free variable?

We could choose either x_1, or x_2 as our second free variable.

The solution set can then be written in one of two ways:

Either as all vectors $\begin{bmatrix} x_1 \\ 6 - 2x_1 \\ 0 \\ x_4 \end{bmatrix}$ **or** as all vectors $\begin{bmatrix} 3 - 0.5x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix}$

Don't choose too many free variables!

Question L13.6: Why would it be wrong to write the solution set

in the above example as the set of all vectors $\begin{bmatrix} 3 - 0.5x_2 \\ 6 - 2x_1 \\ 0 \\ x_4 \end{bmatrix}$?

Recall that what makes variables "free" is that they can take any combination of values whatsoever. For example, in the above we could simply set $x_1 = x_2 = x_4 = 0$, which would give us the vector

$$\begin{bmatrix} 3 - 0.5x_2 \\ 6 - 2x_1 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

But this vector is not in the solution set.