Lecture 14: Solving Systems of Linear Equations by Gaussian Elimination, Part I

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MATH3200: Applied Linear Algebra

Review: The extended matrix

Consider a system of linear equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

The *extended* aka *augmented* matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ represents this system:

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

Equivalent systems and their augmented matrices

We will say that two systems of m linear equations in n variables are *equivalent* if they have the same sets of solutions.

Similarly, we will say that two matrices $[\mathbf{A}_1, \vec{\mathbf{b}}_1], [\mathbf{A}_2, \vec{\mathbf{b}}_2]$ of order $m \times (n+1)$ are *equivalent* if they represent equivalent systems of linear equations.

Here a matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ is said to *represent* a system of linear equations if it is the augmented matrix of the system.

Note that any matrix with at least two columns represents some system of linear equations.

Solving a system boils down to transforming it successively into simpler equivalent systems.

This can be accomplished by successively transforming extended matrices into equivalent ones.

Review: A strategy for doing these transformations

When the extended matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & & \dots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

of the system is in *row echelon form*, we can accomplish this by back-substitution.

If not, then we can transform the system step-by-step into an *equivalent system*, that is, a system with the same solution set, whose extended matrix is in row echelon form.

Each step involves an *elementary operation on systems of linear equations* that does not change the solution set.

Review: Elementary operations on systems of linear equations

Consider a system of m linear equations in n variables.

Each of the following operations preserves the set of solutions and thus transforms the system into an equivalent one:

(i) Interchanging the positions of any two equations.

(ii) Multiplying an equation by a nonzero scalar.

(iii) Adding to one equation a scalar multiple of another equation.

In practice, it is more convenient to work with *elementary row operations on extended matrices.*

Elementary row operations on matrices

Consider a matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ of order $m \times (n+1)$.

Each of the following *elementary row operations* transforms $[\mathbf{A}, \vec{\mathbf{b}}]$ into an equivalent matrix:

(E1) Interchanging any two rows.

(E2) Multiplying any row by a nonzero scalar.

(E3) Adding to one row of the matrix a scalar times another row of the matrix.

The method of Gaussian elimination

Gaussian elimination is a method for solving systems of linear equations. It is named after Carl Friedrich Gauss (1777–1855).

The method relies on transforming the augmented matrix of a given system into an equivalent matrix in row echelon form by successive elementary row operations.

Gaussian elimination usually works well for numerical solutions on the computer.

We will see later that it also gives us some additional insights.

Where do we want to go?
Review of the row echelon form

A matrix is in *row echelon form,* or simply a *echelon form* if:

(R1) All *zero rows* appear below all *nonzero rows* when both types are present.

(R2) The first nonzero entry in any nonzero row is 1.

(R3) All elements in the same column below the first nonzero element of a nonzero row are 0.

(R4) The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.

The method of Gaussian elimination: Example 1

Consider the following system of linear equations:

$$\begin{aligned} 0.5x_1 - x_2 + x_3 &= 2 \\ 2x_1 - 7x_2 - 3x_3 &= 0 \\ x_1 + x_2 + x_3 &= 5 \end{aligned}$$

The augmented matrix is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 0.5 & -1 & 1 & 2 \\ 2 & -7 & -3 & 0 \\ 1 & 1 & 1 & 5 \end{bmatrix}$$

Ideally, we'd like to get an equivalent matrix that looks like this:

$$\begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$$

Gaussian elimination for Example 1

We want to transform our matrix into: $\begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$

In order to keep track of our work, we always indicate in shorthand notation which elementary row operations are used at each step.

$$\begin{bmatrix} 0.5 & -1 & 1 & 2 \\ 2 & -7 & -3 & 0 \\ 1 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R1 \mapsto 2R1} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 2 & -7 & -3 & 0 \\ 1 & 1 & 1 & 5 \end{bmatrix}$$

Here we multiplied Row 1 by 2.

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 2 & -7 & -3 & 0 \\ 1 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 2R1} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 1 & 1 & 1 & 5 \end{bmatrix}$$

Here we subtracted two times Row 1 from Row 2.

Gaussian elimination for Example 1, continued

We want to transform our matrix into: $\begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$

In the next step we want to achieve:

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 1 & 1 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

Question L14.1: Which elementary row operation should we perform?

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 1 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R1} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 0 & 3 & -1 & 1 \end{bmatrix}$$

We need to subtract Row 1 from Row 3.

Gaussian elimination for Example 1, continued

We want to transform our matrix into: $\begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & \color{blue}{0} & 1 & ? \end{bmatrix}$

In the next step we want to achieve:

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 0 & \color{red}{3} & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & \color{blue}{0} & ? & ? \end{bmatrix}$$

Question L14.2: Which elementary row operation should we perform?

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 0 & \color{red}{3} & -1 & 1 \end{bmatrix} \xrightarrow{R3 \mapsto R3+R2} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 0 & \color{blue}{0} & -8 & -7 \end{bmatrix}$$

We need to add Row 2 to Row 3.

Gaussian elimination for Example 1, continued

We want to transform our matrix into: $\begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$

In the next step we want to achieve:

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 0 & 0 & -8 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & ? & ? \end{bmatrix}$$

Question L14.3: Which elementary row operation should we perform?

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & -3 & -7 & -8 \\ 0 & 0 & -8 & -7 \end{bmatrix} \xrightarrow{R2 \mapsto R2/(-3)} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & -8 & -7 \end{bmatrix}$$

We need to divide Row 2 by -3.

Gaussian elimination for Example 1, completed

We want to transform our matrix into: $\begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$

In the next step we want to achieve:

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & -8 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$$

Question L14.4: Which elementary row operation should we perform?

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & -8 & -7 \end{bmatrix} \xrightarrow{R3 \mapsto R3/(-8)} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 1 & 7/8 \end{bmatrix}$$

We need to divide Row 3 by -8.

Example 1: Back-substitution

We have transformed the augmented matrix of the original system into an equivalent matrix in row echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 1 & 7/8 \end{bmatrix}$$

It represents the following equivalent system:

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 4 \\ x_2 + \frac{7}{3}x_3 &= \frac{8}{3} \\ x_3 &= \frac{7}{8} \end{aligned}$$

Back-substitution gives the solution: $x_3 = \frac{7}{8},\ x_2 = \frac{5}{8},\ x_1 = \frac{7}{2}$.

The ideal order of operations

Although there may be many different ways to successfully perform Gaussian elimination on a given matrix, there is one ideal order of steps that often works best, as long as it it feasible. Here is an illustration of it for a 3×4 matrix. For each step, elements of the matrix that may change are shown in color. You will see that this order of operations preserves zeros and ones that were created at earlier steps of the process.

Start with dividing the first row by a_{11}:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \xrightarrow{R1 \mapsto R1/a_{11}} \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Question L14.5: Is this step always feasible?

No. This step is feasible only if $a_{11} \neq 0$; otherwise we may need to switch some rows first.

The ideal order of operations, continued

Now use the 1 in the upper left corner as a so-called *pivot* to cancel the elements of the matrix below it:

$$\begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \xrightarrow{R2 \mapsto R2 - a_{21} R1} \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$\begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \xrightarrow{R3 \mapsto R3 - a_{31} R1} \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \end{bmatrix}$$

Question L14.6: Are the two steps shown on this slide always feasible?

Yes.

The ideal order of operations, completed

If feasible, divide the second row by b_{22} and use the resulting 1 as a pivot to cancel the element(s) below it. For larger matrices, continue analogously until you get the row echelon form.

$$\begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \end{bmatrix} \xrightarrow{R2 \mapsto R2/b_{22}} \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & b_{32} & b_{33} & b_{34} \end{bmatrix}$$

$$\begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & b_{32} & b_{33} & b_{34} \end{bmatrix} \xrightarrow{R3 \mapsto R3 - b_{32} R2} \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & c_{33} & c_{34} \end{bmatrix}$$

$$\begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & c_{33} & c_{34} \end{bmatrix} \xrightarrow{R3 \mapsto R3/c_{33}} \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & d_{34} \end{bmatrix}$$

Example 2 of a Gaussian elimination

Recall that in Conversation 13, Bob transformed the extended matrix of the system

$$\begin{array}{ccccccc} x_a & + & x_b & + & x_c & = & 11 \\ x_a & & & - & x_c & = & -1 \\ x_a & - & 2x_b & & & = & 0 \end{array}$$

into row echelon form. He did in fact perform a Gaussian elimination. To see this, let us trace his steps, in the ideal order.

Here the extended matrix of the system is:

$$\begin{bmatrix} 1 & 1 & 1 & 11 \\ 1 & 0 & -1 & -1 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$

It has already a 1 in the upper left corner.

Example 2 of a Gaussian elimination, continued

Now use the 1 in the upper left corner as a pivot to cancel the elements of the matrix below it:

$$\begin{bmatrix} 1 & 1 & 1 & 11 \\ 1 & 0 & -1 & -1 \\ 1 & -2 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - R1} \begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & -1 & -2 & -12 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & -1 & -2 & -12 \\ 1 & -2 & 0 & 0 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R1} \begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & -1 & -2 & -12 \\ 0 & -3 & -1 & -11 \end{bmatrix}$$

Example 2 of Gaussian elimination, completed

Next divide the second row by -1 and use the resulting 1 as a pivot to cancel the element below it. In the last step, divide the third row by 5 to obtain 1 as the first nonzero element of this row.

$$\begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & -1 & -2 & -12 \\ 0 & -3 & -1 & -11 \end{bmatrix} \xrightarrow{R2 \mapsto R2/(-1)} \begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & 1 & 2 & 12 \\ 0 & -3 & -1 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & 1 & 2 & 12 \\ 0 & -3 & -1 & -11 \end{bmatrix} \xrightarrow{R3 \mapsto R3+3R2} \begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 5 & 25 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 5 & 25 \end{bmatrix} \xrightarrow{R3 \mapsto R3/5} \begin{bmatrix} 1 & 1 & 1 & 11 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Take-home message

- *Gaussian elimination* is a method for transforming matrices into *row echelon form* by successively applying *elementary row operations.*
- If we start with the augmented matrix of a linear system, the method will transform it into the extended matrix of an *equivalent system* that has the same solution set.
- The ideal order of operations creates ones and zeros in successive columns, starting from the leftmost column. In each column, it is usually most convenient to first create a 1 in the appropriate place and then use it as a *pivot* to cancel the elements below it.
- After you have obtained an equivalent matrix in row echelon form, translate it back into a system of linear equations and solve that equivalent system with back-substitution.