Lecture 15: Solving Systems of Linear Equations by Gaussian Elimination, Part II

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MATH3200: Applied Linear Algebra

Review: Elementary row operations on matrices

Consider a matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ of order $m \times (n+1)$.

Each of the following *elementary row operations* transforms $[\mathbf{A}, \vec{\mathbf{b}}]$ into an equivalent matrix:

(E1) Interchanging any two rows.

(E2) Multiplying any row by a nonzero scalar.

(E3) Adding to one row of the matrix a scalar times another row of the matrix.

Review: Row echelon form and Gaussian elimination

A matrix is in *row echelon form* or simply *echelon form* if:

(R1) All zero rows appear below all nonzero rows when both types are present.

(R2) The first nonzero entry in any nonzero row is 1.

(R3) All elements in the same column below the first nonzero element of a nonzero row are 0.

(R4) The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.

The goal of Gaussian elimination is to transform the augmented matrix into an equivalent one in row echelon form by using a sequence of elementary row operations.

Pivots

Whenever one element in a matrix is used to *cancel* another element to zero by elementary row operation (E3), the first element is called the *pivot.*

For example, in

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 2 & 5 & 6 & 4 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 2R1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 0 & 3 & 4 & 6 \end{bmatrix}$$

the pivot is the 1 in the upper left corner.

In numerical implementations of Gaussian elimination it is important to choose the pivots judiciously to minimize the effect of rounding errors. Details of how to do this are being taught in our courses on numerical methods.

The order of operations

In the previous lecture you have seen an "ideal order of operations" that *often* works best when performing Gaussian elimination.

In this order of operations, zeros are created in the appropriate places column by column, starting from the leftmost column. This part of the strategy should *always* be followed.

In the ideal order, you would start your work on column k with dividing row number k by a_{kk}. This is somewhat optional. In this lecture we will study some examples where it may be more convenient, or even necessary, to start your work on column k in a different way.

Example 1

Consider the following system of linear equations:

$$
\begin{aligned}
3x_1 - 3x_2 + 3x_3 &= 6 \\
2x_1 + x_2 - 3x_3 &= 0 \\
3x_1 - 2x_3 &= 5
\end{aligned}
$$

The augmented matrix is

$$
[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 3 & -3 & 3 & 6 \\ 2 & 1 & -3 & 0 \\ 3 & 0 & -2 & 5 \end{bmatrix}
$$

Gaussian elimination for Example 1

Here it would be more convenient to use 3 directly as a pivot for canceling the 3 in the lower left corner:

$$\begin{bmatrix} 3 & -3 & 3 & 6 \\ 2 & 1 & -3 & 0 \\ 3 & 0 & -2 & 5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R1} \begin{bmatrix} 3 & -3 & 3 & 6 \\ 2 & 1 & -3 & 0 \\ 0 & 3 & -5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 3 & 6 \\ 2 & 1 & -3 & 0 \\ 0 & 3 & -5 & -1 \end{bmatrix} \xrightarrow{R1 \mapsto R1/3} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 0 \\ 0 & 3 & -5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 0 \\ 0 & 3 & -5 & -1 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 2R1} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -5 & -4 \\ 0 & 3 & -5 & -1 \end{bmatrix}$$

Gaussian elimination for Example 1, continued

Again, it is easier to work with 3 as a pivot for the next step:

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -5 & -4 \\ 0 & 3 & -5 & -1 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R2} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -5 & -4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -5 & -4 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{R2 \mapsto R2/3} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -5/3 & -4/3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -5/3 & -4/3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{R3 \mapsto R3/3} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -5/3 & -4/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1: Reading off the solution

We have transformed the augmented matrix of the original system into an equivalent matrix in row echelon form:

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -5/3 & -4/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It represents the following equivalent system:

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_2 + \frac{-5}{3}x_3 &= \frac{-4}{3} \\ 0 &= 1 \end{aligned}$$

Question L15.1: What can we say about the solution set?

This system is *inconsistent.* We conclude that the original system is also inconsistent and has no solution.

Example 2

Consider the following system of linear equations:

$$\begin{aligned} 3x_2 + 4x_3 &= 6 \\ x_1 + x_2 + x_3 &= -1 \\ 2x_1 + 5x_2 + 6x_3 &= 4 \end{aligned}$$

The augmented matrix is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 0 & 3 & 4 & 6 \\ 1 & 1 & 1 & -1 \\ 2 & 5 & 6 & 4 \end{bmatrix}$$

Question L15.2: What should we do here as a first step?

Switch rows 1 and 2.

Gaussian elimination for Example 2

$$\begin{bmatrix} 0 & 3 & 4 & 6 \\ 1 & 1 & 1 & -1 \\ 2 & 5 & 6 & 4 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 2 & 5 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 2 & 5 & 6 & 4 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 2R1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 0 & 3 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 0 & 3 & 4 & 6 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R2/3} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 4/3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The resulting matrix is in row echelon form.

Example 2: Finding the solution set

We have transformed the augmented matrix of the original system

into an equivalent matrix in row echelon form: $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 4/3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

It represents the following equivalent system:

$$\begin{aligned} x_1 + x_2 + x_3 &= -1 \\ x_2 + \frac{4}{3}x_3 &= 2 \\ 0 &= 0 \end{aligned}$$

Question L15.3: What can we say about the solution set?

This system is *underdetermined.* If we choose x_3 as our free variable, we find by back-substitution that the solution set consists

of all vectors of the form $\begin{bmatrix} -3 + (1/3)x_3 \\ 2 - (4/3)x_3 \\ x_3 \end{bmatrix}$

Example 3

Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & + & 2x_5 & = & 0 \\ & & & & & & 3x_4 & + & 3x_5 & = & 6 \\ 2x_1 & & & + & 3x_3 & & & & & = & -6 \\ & & x_2 & & & - & 4x_4 & - & x_5 & = & 1 \end{array}$$

The augmented matrix is

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 3 & 6 \\ 2 & 0 & 3 & 0 & 0 & -6 \\ 0 & 1 & 0 & -4 & -1 & 1 \end{bmatrix}$$

Note that this system has 5 variables, but only 4 equations.

Question L15.4: What can we say about the solution set based on this observation?

Gaussian elimination for Example 3

Since the number of variables exceeds the number of equations, the system cannot have a unique solution. It can be either underdetermined or inconsistent.

We start Gaussian elimination:

$$\begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 3 & 6 \\ 2 & 0 & 3 & 0 & 0 & -6 \\ 0 & 1 & 0 & -4 & -1 & 1 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R1} \begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 3 & 6 \\ 0 & -1 & 0 & 1 & -2 & -6 \\ 0 & 1 & 0 & -4 & -1 & 1 \end{bmatrix}$$

Question L15.5: What should we do as the next step?

Switch Row 2 and Row 4:

$$\begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 3 & 6 \\ 0 & -1 & 0 & 1 & -2 & -6 \\ 0 & 1 & 0 & -4 & -1 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R4} \begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 \\ 0 & -1 & 0 & 1 & -2 & -6 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{bmatrix}$$

Gaussian elimination for Example 3, completed

$$\begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 \\ 0 & -1 & 0 & 1 & -2 & -6 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{bmatrix} \xrightarrow{R3 \mapsto R3 + R2} \begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 \\ 0 & 0 & 0 & -3 & -3 & -5 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 \\ 0 & 0 & 0 & -3 & -3 & -5 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{bmatrix} \xrightarrow{R4 \mapsto R4 + R3} \begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 \\ 0 & 0 & 0 & -3 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Question L15.6: What can we say about the solution set of this system?

Example 3: This system is inconsistent

We have transformed the augmented matrix of the original system into an equivalent matrix in row echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & -1 & 2 & 0 \\ 0 & 1 & 0 & -4 & -1 & 1 \\ 0 & 0 & 0 & -3 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It represents the following equivalent system:

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & + & 2x_5 & = & 0 \\ & & x_2 & & & - & 4x_4 & - & x_5 & = & 1 \\ & & & & & & -3x_4 & - & 3x_5 & = & -5 \\ & & & & & & & & 0 & = & 1 \end{array}$$

This system is *inconsistent.*

Example 4

Consider the following system of linear equations:

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 6 \\ x_1 & & & = & 1 \end{array}$$

Here can immediately see that the only solution has coordinates $x_1 = 1$ and $x_2 = 4$.

But the augmented matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

is not in row echelon form.

One might be tempted to transform this matrix into row echelon form by switching columns instead of rows, but this is *not permitted in Gaussian elimination.*

To see why, let's try it and see what would happen if we did.

Example 4: What would happen with column operations?

When we switch Columns 1 and 2 we get:

$$\begin{bmatrix} 2 & 1 & 6 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{C1 \leftrightarrow C2} \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

The resulting matrix is in row echelon form and is the extended matrix of the system

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 6 \\ & & x_2 & = & 1 \end{array}$$

The solution of this system has coordinates $x_1 = 4$ and $x_2 = 1$.

Question L15.7: Did we get the same solution as for the original system?

No. The solution of the original system has coordinates $x_1 = 1$ and $x_2 = 4$. These coordinates got switched by the column operation.

We can see that the operation on the columns did *not* preserve the solution set; it did *not* give an equivalent matrix.

Take-home message

When performing Gaussian elimination to solve a linear system, *only* use the elementary row operations (E1), (E2), (E3).

In particular, you are *not allowed* to use column operations instead of row operations.

Perform the operations in such an order that you get the right type of entries for the row echelon form in successive columns, starting from the leftmost column.

There is some flexibility in the order in which you work for transforming a given column. In particular, you may need to switch some rows to get usable pivots, and you may want to work with pivots that make the calculations convenient.