Lecture 16: Introduction to Inverse Matrices

Winfried Just
Department of Mathematics, Ohio University

MATH3200: Applied Linear Algebra

The definition of the matrix inverse

Let $\mathbf{A}$ be an $n \times n$ square matrix.

The *inverse of* $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.

Theorem

The inverse $\mathbf{A}^{-1}$*, if it exists, is unique and satisfies* $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$*.*

Note that the inverse of a matrix is the analogue of a reciprocal

$a^{-1} = \frac{1}{a}$ of a number.

Note that the reciprocal $\frac{1}{a}$ of a number

exists if, and only if, $a \neq 0$*.*

An example of an inverse matrix

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

I claim that $\mathbf{B}$ is the inverse matrix of the matrix $\mathbf{A}$.

Question L16.1: How could we convince ourselves that this claim is actually true?

We need to multiply these matrices and check whether the product is the identity matrix $\mathbf{I}_2$.

Let $\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & ? \\ ? & ? \end{bmatrix}$

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When we switch the order of multiplication:

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We can see that $\mathbf{B} = \mathbf{A}^{-1}$ and $\mathbf{A} = \mathbf{B}^{-1}$.

We can also see that:

Verifying whether a given pair of matrices are inverses of each other is easy.

You just need to multiply them and check whether the product is an identity matrix $\mathbf{I}$.

Invertible *vs.* non-invertible matrices

If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}$ is called *invertible* or *non-singular.*
If not, then $\mathbf{A}$ is called *non-invertible* aka *singular.*

Question L16.2: Can you give an example of a singular matrix?

Any square zero matrix $\mathbf{O}_{n\times n}$ is singular.

But these are not the only examples.

Example 1: Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Then $\mathbf{A}$ is not a zero matrix, but we will show that no 2×2 matrix $\mathbf{C}$ can be the inverse of $\mathbf{A}$.

Consider any 2×2 matrix $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

Then $\mathbf{AC} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ 0 & \mathbf{0} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{1} \end{bmatrix} = \mathbf{I}_2.$

No matrix $\mathbf{C}$ can be the inverse matrix $\mathbf{A}^{-1}$, thus $\mathbf{A}$ is singular.

Another example of a non-invertible matrix

Example 2: Let $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ Consider $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

Then $\mathbf{BC} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11} + 2c_{21} & c_{12} + 2c_{22} \\ 3c_{11} + 6c_{21} & 3c_{12} + 6c_{22} \end{bmatrix}$

Question L16.3: Why can this matrix product not be equal to the identity matrix $\mathbf{I}_2$?

Because the second row of **BC** is 3 times its first row, while the

second row of $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not a scalar multiple of the first.

Thus no matrix **C** can be the inverse matrix $\mathbf{B}^{-1}$.

Neither of the singular matrices $\mathbf{A}, \mathbf{B}$ of Examples 1 and 2 is a zero matrix **O**.

Inverses of diagonal matrices

Consider a diagonal matrix of order $n \times n$:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

If $\lambda_i \neq 0$ for all $i = 1, 2, \dots, n$, then $\mathbf{D}$ is invertible and

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda_n} \end{bmatrix}$$

Why does this work?

The product of any two $n \times n$ diagonal matrices is:

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \kappa_1 & 0 & \dots & 0 \\ 0 & \kappa_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \kappa_n \end{bmatrix} = \begin{bmatrix} \lambda_1\kappa_1 & 0 & \dots & 0 \\ 0 & \lambda_2\kappa_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n\kappa_n \end{bmatrix}$$

In particular:

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda_n} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1}{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{\lambda_2}{\lambda_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{\lambda_n}{\lambda_n} \end{bmatrix} = \mathbf{I}_n$$

When is a diagonal matrix singular?

Consider a diagonal matrix of order $n \times n$:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Question L16.4: When is $\mathbf{D}$ non-invertible (singular)?

If $\lambda_i = 0$ for *at least one* $i = 1, 2, \dots, n$.

Then $\mathbf{D}^{-1}$ does not exist.

So far so good . . .

We have seen that

- It is *easy to verify* that a given pair of matrices are inverses of each other (multiply them and check whether the product is an identity matrix $\mathbf{I}$).
- It can be *relatively easy* to find $\mathbf{A}^{-1}$ when $\mathbf{A}$ is a diagonal matrix.
- But how do we find $\mathbf{A}^{-1}$ *in general?*

  Even for a seemingly simple matrix like

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

this seems *hard.*

Take-home message

The *inverse* $\mathbf{A}^{-1}$ of a square matrix $\mathbf{A}$ is a matrix that satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

If the inverse $\mathbf{A}^{-1}$ exists, is unique and satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$.

Only square matrices can have inverses, but many so-called *non-invertible* or *singular* square matrices do not.

If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}$ is called *invertible* or *non-singular.*

A diagonal matrix $\mathbf{D}$ is invertible if, and only if, all diagonal elements are nonzero. In this case, $\mathbf{D}^{-1}$ is the diagonal matrix that has the reciprocals of the diagonal elements of $\mathbf{D}$ on the (main) diagonal.

Except for special cases, such as diagonal matrices, computing the inverse of a square matrix requires a special method.