Lecture 17: Using the Inverse Matrix to Solve Linear Systems

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MATH3200: Applied Linear Algebra

Review: The definition of the matrix inverse

Let $\mathbf{A}$ be an $n \times n$ square matrix.

The *inverse of* $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.

Theorem

The inverse $\mathbf{A}^{-1}$, if it exists, is unique and satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$.

Note that the inverse of a matrix is the analogue of a reciprocal $a^{-1} = \frac{1}{a}$ of a number.

A square matrix $\mathbf{A}$ without an inverse $\mathbf{A}^{-1}$ is called *non-invertible* or *singular.*

If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}$ is *invertible* or *non-singular.*

Linear equations for numbers and matrices

Numbers: Consider a linear equation $ax = b$.

- If $a = 1$, then $x = b$ is the unique solution.
- *If $a \neq 0$,* then $a^{-1}ax = 1x = a^{-1}b$ so that $x = a^{-1}b = \frac{b}{a}$ is the unique solution.
- *If $a = 0$,* then there may be infinitely many solutions or none.

Matrices: Consider a linear equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\mathbf{A}$ is square.

- If $\mathbf{A} = \mathbf{I}$, then $\mathbf{I}\vec{\mathbf{x}} = \vec{\mathbf{x}} = \vec{\mathbf{b}}$ is the unique solution.
- If $\mathbf{A}$ *is invertible,* then $\mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{x}} = \mathbf{I}\vec{\mathbf{x}} = \vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$ so that $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$ is the unique solution.
- If $\mathbf{A}$ *is non-invertible,* then the system is either underdetermined or inconsistent.

Solving a system with the help of $\mathbf{A}^{-1}$: An example

Consider the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of linear equations

$$\begin{aligned} 3x_1 \quad\quad &= 15 \\ 0.5x_2 &= -1 \end{aligned}$$

The coefficient matrix is $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$

Question L17.1: What is $\mathbf{A}^{-1}$?

Since $\mathbf{A}$ is a diagonal matrix, we already know how to calculate

its inverse: $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}$

The unique solution of the above system can be calculated as:

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}^{-1}\vec{\mathbf{b}} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Solving a system with the help of $\mathbf{A}^{-1}$: A second example

Consider the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of linear equations

$$x_1 + 2x_2 = 5$$
$$3x_1 + 4x_2 = -1$$

The coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

In the previous lecture we found its inverse $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

Question 17.2: How would you use $\mathbf{A}^{-1}$ to solve the above system?

The unique solution can be obtained as:

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}^{-1}\vec{\mathbf{b}} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -11 \\ 8 \end{bmatrix}$$

Take-home message

Let $\mathbf{A}$ be a square matrix and let $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ be a system of linear equations with coefficient matrix $\mathbf{A}$.

When $\mathbf{A}^{-1}$ exists and is known, then the linear system $\mathbf{A} = \vec{\mathbf{b}}$ has a unique solution that can be computed as the product $\mathbf{A}^{-1}\vec{\mathbf{b}}$.

When $\mathbf{A}^{-1}$ does not exist, then the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is either underdetermined or inconsistent.