Lecture 18: Finding Inverse Matrices by Gauss-Jordan Elimination

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MATH3200: Applied Linear Algebra

Review: The definition of the matrix inverse

Let $\mathbf{A}$ be an $n \times n$ square matrix.

The *inverse of* $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.

Theorem

The inverse $\mathbf{A}^{-1}$, if it exists, is unique and satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$.

When $\mathbf{A}$ is a zero matrix, then $\mathbf{A}$ does not have an inverse.

For $n > 1$, there are infinitely many more examples of *non-invertible* aka *singular* matrices $\mathbf{A}$ of order $n \times n$, that is matrices that do not have an inverse matrix $\mathbf{A}^{-1}$.

If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}$ is *invertible* or *non-singular.*

How to find inverse matrices?

We have see that:

- It is *easy to verify* that a given pair of matrices are inverses of each other (multiply them and check whether the product is an identity matrix $\mathbf{I}$).
- It can be *relatively easy* to find $\mathbf{A}^{-1}$ when $\mathbf{A}$ is a diagonal matrix.
- But how do we find $\mathbf{A}^{-1}$ *in general?*

  Even for a seemingly simple matrix like

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

  this seems *hard.*

An observation

An $n \times n$ matrix $\mathbf{A}$ can have an inverse only if Gaussian elimination produces a matrix with all diagonal elements equal to 1.

For $n = 3$ this looks as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \overset{\text{Gaussian elimination}}{\longrightarrow} \begin{bmatrix} 1 & ? & ? \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can keep going and apply elementary row operation (E3) more times until we get $\mathbf{I}$:

$$\begin{bmatrix} 1 & ? & ? \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{bmatrix} \overset{\text{More applications of (E3)}}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So what? How could this help?

A magic trick: Gauss-Jordan elimination

Let $\mathbf{A}$ be an $n \times n$ matrix. Form an $n \times 2n$ matrix $\mathbf{C}$ by dropping the internal brackets in $[\mathbf{A}, \mathbf{I}_n]$ and replacing them with a vertical dividing line for visual clarity. For $n = 3$ we get:

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Perform Gaussian elimination. If *the first half* of the resulting matrix in row-echelon form has a zero row, then $\mathbf{A}$ is not invertible. *Otherwise* keep going and apply instances of (E3) until the first half turns into $\mathbf{I}_n$ so that the entire matrix is in reduced row echelon form. For $n = 3$ the result will look like:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right]$$

Let's see what we get for the matrix B in the second half.

Trying out the magic trick

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Here we already know $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

Form a 2×4 matrix $\mathbf{C}$ and perform Gaussian elimination on it:

$$\mathbf{C} = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R2 \mapsto R2 - 3R1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{R2 \mapsto R2/(-2)} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 \end{array}\right]$$

Apply (E3) one more time to turn the first half into $\mathbf{I}_2$:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 \end{array}\right] \xrightarrow{R1 \mapsto R1 - 2R2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1.5 & -0.5 \end{array}\right]$$

Magically, the matrix B in the right half is $\mathbf{A}^{-1}$!

Trying the magic trick on another matrix

Let $\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$ Here we don't know $\mathbf{A}^{-1}$.

Form a 3×6 matrix $\mathbf{C}$ and perform Gaussian elimination on it.

Start by subtracting row 1 from row 2:

$$\mathbf{C} = \left[\begin{array}{ccc|ccc} 0.5 & 0.5 & 0 & 1 & 0 & 0 \\ {\color{red}0.5} & 0 & 0.5 & 0 & 1 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0.5 & 0.5 & 0 & 1 & 0 & 0 \\ {\color{blue}0} & -0.5 & 0.5 & -1 & 1 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 1 \end{array}\right]$$

Next add row 2 to row 3:

$$\left[\begin{array}{ccc|ccc} 0.5 & 0.5 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0.5 & -1 & 1 & 0 \\ 0 & {\color{red}0.5} & 0.5 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0.5 & 0.5 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0.5 & -1 & 1 & 0 \\ 0 & {\color{blue}0} & 1 & -1 & 1 & 1 \end{array}\right]$$

Trying the magic trick on another matrix, continued

Multiply row 1 by 2:

$$\left[\begin{array}{ccc|ccc} \color{red}{0.5} & 0.5 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0.5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} \color{blue}{1} & 1 & 0 & 2 & 0 & 0 \\ 0 & -0.5 & 0.5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Multiply row 2 by -2:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & \color{red}{-0.5} & 0.5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & \color{blue}{1} & -1 & 2 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

The first half is now in row echelon form.

Question L18.1: Are we done?

Not yet. We still need to get rid of the nonzero off-diagonal elements in the first half of this matrix.

Trying the magic trick on another matrix, completed

Add row 3 to row 2:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & \color{red}{-1} & 2 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & \color{blue}{0} & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Question L18.2: What should we do next?

Subtract row 2 from row 1:

$$\left[\begin{array}{ccc|ccc} 1 & \color{red}{1} & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & \color{blue}{0} & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Did the magic work? We will verify this by hand.

Why does the magic trick work?

This magic trick is often called *Gauss-Jordan elimination.*

To see why it works, it will be most convenient to treat the matrix $[\mathbf{A}, \mathbf{I}_n]$ and the matrix $\mathbf{C}$ that is obtained from it after dropping the internal brackets as the same object (shown for $n = 3$):

$$\mathbf{C} = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] = [\mathbf{A}, \mathbf{I}_n]$$

The end result of the procedure can be written as follows (first the special case for $n = 3$, then the general case is shown):

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right] = [\mathbf{I}_n, \mathbf{B}]$$

Why does the magic trick work?

The procedure involves successively applying elementary row operations to the matrix $\mathbf{C}$. These can be implemented by successively multiplying $\mathbf{C}$ from the left by elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \ldots, \mathbf{E}_k$:

$\mathbf{C} = [\mathbf{A}, \mathbf{I}_n]$

Question L18.3: What does multiplying $\mathbf{C}$ from the left with $\mathbf{E}_1$ do to the matrices $\mathbf{A}$ and $\mathbf{I}_n$?

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$$\mathbf{E}_1\mathbf{C} = \mathbf{E}_1[\mathbf{A}, \mathbf{I}_n]$$

These operations do *the same thing* to the two matrices in the expression on the right:

$$[\mathbf{E}_1\mathbf{A}, \quad \mathbf{E}_1\mathbf{I}_n]$$

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$$\mathbf{E}_2\mathbf{E}_1\mathbf{C} = \mathbf{E}_2\mathbf{E}_1[\mathbf{A}, \mathbf{I}_n]$$

These operations do *the same thing* to the two matrices in the expression on the right:

$$[\mathbf{E}_2\mathbf{E}_1\mathbf{A}, \quad \mathbf{E}_2\mathbf{E}_1\mathbf{I}_n]$$

Why does the magic trick work?

The procedure involves successively applying elementary row operations to the matrix $\mathbf{C}$. These can be implemented by successively multiplying $\mathbf{C}$ from the left by elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \dots, \mathbf{E}_k$:

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{C} = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1[\mathbf{A}, \mathbf{I}_n]$$

These operations do *the same thing* to the two matrices in the expression on the right:

$$[\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}, \quad \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{I}_n]$$

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$$\dots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{C} = \dots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1[\mathbf{A}, \mathbf{I}_n]$$

These operations do *the same thing* to the two matrices in the expression on the right:

$$[\dots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}, \quad \dots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{I}_n]$$

Why does the magic trick work?

The procedure involves successively applying elementary row operations to the matrix $\mathbf{C}$. These can be implemented by successively multiplying $\mathbf{C}$ from the left by elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \ldots, \mathbf{E}_k$:

$$\mathbf{E}_k \ldots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{C} = \mathbf{E}_k \ldots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1[\mathbf{A}, \mathbf{I}_n]$$

These operations do *the same thing* to the two matrices in the expression on the right:

$$[\mathbf{E}_k \ldots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}, \quad \mathbf{E}_k \ldots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{I}_n] = [\mathbf{I}_n, \mathbf{B}]$$

The second coordinate shows that

$$\mathbf{B} = \mathbf{E}_k \ldots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{I}_n = \mathbf{E}_k \ldots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1.$$

The first coordinate shows that $(\mathbf{E}_k \ldots \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)\mathbf{A} = \mathbf{B}\mathbf{A} = \mathbf{I}_n$. It follows that $\mathbf{B} = \mathbf{A}^{-1}$.

Practice: One more example of Gauss-Jordan elimination

Let $\mathbf{A} = \begin{bmatrix} 0 & 0.5 \\ 1 & 2 \end{bmatrix}$ We want to find $\mathbf{A}^{-1}$.

Form a 2×4 matrix $\mathbf{C} = \left[\begin{array}{cc|cc} 0 & 0.5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$.

Question L18.4: What should we do next?

Switch rows 1 and 2:

$$\left[\begin{array}{cc|cc} \color{red}{0} & 0.5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|cc} \color{blue}{1} & 2 & 0 & 1 \\ 0 & 0.5 & 1 & 0 \end{array}\right]$$

Question L18.5: What should we do next?

Multiply row 2 by 2:

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & \color{red}{0.5} & 1 & 0 \end{array}\right] \xrightarrow{R2 \mapsto 2R2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & \color{blue}{1} & 2 & 0 \end{array}\right]$$

Practice example of Gauss-Jordan elimination, completed

Let $\mathbf{A} = \begin{bmatrix} 0 & 0.5 \\ 1 & 2 \end{bmatrix}$ We want to find $\mathbf{A}^{-1}$.

We formed a 2×4 matrix $\mathbf{C} = \left[\begin{array}{cc|cc} 0 & 0.5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$

and transformed it into $\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right]$

Question L18.6: What should we do next?

Subtract 2 times row 2 from row 1:

$$\left[\begin{array}{cc|cc} 1 & \color{red}{2} & 0 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R2 \mapsto R1 - 2R2} \left[\begin{array}{cc|cc} 1 & \color{blue}{0} & -4 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

The matrix $\mathbf{B} = \begin{bmatrix} -4 & 1 \\ 2 & 0 \end{bmatrix}$ in the right half is $\mathbf{A}^{-1}$

Summary: A magic trick aka Gauss-Jordan elimination

Let $\mathbf{A}$ be an $n \times n$ matrix. Form an $n \times 2n$ matrix $\mathbf{C}$ by dropping the internal brackets in $[\mathbf{A}, \mathbf{I}_n]$ and replacing them with a vertical dividing line for visual clarity. For $n = 3$ we get:

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Perform Gaussian elimination. If *the first half* of the resulting matrix in row-echelon form has a zero row, then $\mathbf{A}$ is not invertible. *Otherwise* keep going and apply instances of (E3) until the first half turns into $\mathbf{I}_n$ so that the entire matrix is in reduced row echelon form. For $n = 3$ the result will look like:

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The matrix B in the second half will be the inverse $\mathbf{A}^{-1}$.