Lecture 19: Properties of the Inverse of a Matrix

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MATH3200: Applied Linear Algebra

Inverses of matrix products

Theorem

Let $\mathbf{A}, \mathbf{B}, \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ *be invertible matrices of the same order. Then*

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

$(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{k-1}\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1} \ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$.

Proof: We prove the second part. The first part is a special case.

Consider the product

$(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{k-1}\mathbf{A}_k)(\mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1} \ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1})$

Question L19.1: What do we need to show about this product?

We need to show that this product is the identity matrix.

Now group the factors differently:

$\cdots = (\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{k-1})(\mathbf{A}_k\mathbf{A}_k^{-1})(\mathbf{A}_{k-1}^{-1} \ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1})$

Question L19.2: What can we say about the middle term?

The middle term is the identity matrix.

Inverses of matrix products, completed

Theorem

Let $\mathbf{A}, \mathbf{B}, \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ *be invertible matrices of the same order. Then*

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

$$(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{k-1}\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1} \ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}.$$

Proof: We prove the second part. The first part is a special case.

$(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{k-1}\mathbf{A}_k)(\mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1} \ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1})$

$= (\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{k-1})(\mathbf{A}_k\mathbf{A}_k^{-1})(\mathbf{A}_{k-1}^{-1} \ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1})$

$= (\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{k-1})\mathbf{I}(\mathbf{A}_{k-1}^{-1} \ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1})$

$= (\mathbf{A}_1\mathbf{A}_2 \ldots)(\mathbf{A}_{k-1}\mathbf{A}_{k-1}^{-1})(\ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}) = (\mathbf{A}_1\mathbf{A}_2 \ldots)\mathbf{I}(\ldots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1})$

$\ldots$ (We keep going like this.)

$= (\mathbf{A}_1\mathbf{A}_2)(\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}) = \mathbf{A}_1(\mathbf{A}_2\mathbf{A}_2^{-1})\mathbf{A}_1^{-1} = \mathbf{A}_1\mathbf{I}\mathbf{A}_1^{-1} = \mathbf{A}_1\mathbf{A}_1^{-1} = \mathbf{I}.$ □

Some numerical examples

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Then $\mathbf{AB} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

Here $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$ and $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix} = (\mathbf{AB})^{-1}$, as $\begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} = \mathbf{I}$

But $(\mathbf{AB})^{-1} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix} \neq \mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} -2 & -1 \\ 1.5 & 0.5 \end{bmatrix}$

Also, $\mathbf{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $\left(\mathbf{A}^T\right)^{-1} = \begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} = \left(\mathbf{A}^{-1}\right)^T$.

The inverses of a matrix transpose

Theorem

Let $\mathbf{A}$ *be an invertible matrix. Then*
$\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$.

Proof:

Question L19.3: What do we need to show in this proof?

We need to verify that $\left(\mathbf{A}^{-1}\right)^T \mathbf{A}^T = \mathbf{I}$.

Question L19.4: What is the correct formula for $\mathbf{B}^T\mathbf{C}^T$?

Since for any two square matrices $\mathbf{B}, \mathbf{C}$ of the same order we have:
$\mathbf{B}^T\mathbf{C}^T = (\mathbf{CB})^T$, we get:
$\left(\mathbf{A}^{-1}\right)^T \mathbf{A}^T = \left(\mathbf{A}\mathbf{A}^{-1}\right)^T = \mathbf{I}^T = \mathbf{I} \;\square$.

The inverses of upper- and lower-triangular matrices

Recall that an *upper-triangular matrix* is a square matrix $\mathbf{U}$ for which all entries *below* the main diagonal are zero and a *lower-triangular matrix* is a square matrix $\mathbf{L}$ for which all entries *above* the main diagonal are zero.

Note that, in particular, any square matrix in generalized row echelon form is upper-triangular.

Theorem

Let $\mathbf{U}$ *be an invertible upper-triangular matrix. Then* $\mathbf{U}^{-1}$ *is also upper-triangular.*

Let $\mathbf{L}$ *be an invertible lower-triangular matrix. Then* $\mathbf{L}^{-1}$ *is also lower-triangular.*

We will illustrate the proof in Module 31.

Examples of inverses of triangular matrices

For example, both

$$\mathbf{U} = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{U}^{-1} = \begin{bmatrix} 1 & -0.4 \\ 0 & 0.2 \end{bmatrix}$$

are upper-triangular.

Similarly, both

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -10 & 0 \\ 7 & 0 & 0.1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & -0.1 & 0 \\ -70 & 0 & 10 \end{bmatrix}$$

are lower-triangular.

Take-home message

We have seen the following properties of matrix inverses:

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- $\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$.
- An upper- or lower-triangular matrix is invertible if, and only if, all of its diagonal elements are nonzero.
- The inverse of an invertible upper-triangular matrix is also upper-triangular.
- The inverse of an invertible lower-triangular matrix is also lower-triangular.