Lecture 21: Introduction to Linear Combinations

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MATH3200: Applied Linear Algebra

Chapter 3: Key concepts of linear algebra

Now we start Chapter 3. It covers a number of somewhat abstract concepts of linear algebra and is the hardest part of this course.

Knowledge of these concepts will empower you to skillfully use advanced tools of linear algebra.

We will introduce these concepts one by one and show how they relate to each other and to certain applications. Typically, a concept will be introduced in a short lecture, and then we will discuss its meaning, relation to other concepts, and its applications in a dialogue with our protagonists.

In this lecture we will introduce and illustrate the first of these concepts: linear combinations.

Linear combinations: The definition

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients* here, such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

Consider the following sets of vectors:

$A = \{[1,2],[3,4],[0,5]\}$ $\qquad$ $B = \{[1,2],[1,2,3],[1,2,3,4]\}$

$$C = \left\{[1,2],[0,5],\begin{bmatrix}3\\4\end{bmatrix}\right\} \qquad D = \left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-3\\4\end{bmatrix},\begin{bmatrix}5\\-6\end{bmatrix}\right\}$$

Question L21.1: For which of these sets does the above definition make sense so that we can meaningfully form linear combinations?

Only for sets A and D.

The order of all vectors must be the same

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \ldots, d_n$, called *coefficients* here, such that

$$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n.$$

To make addition meaningful, the vectors in the above definition must all be of the same order. They could be either all row vectors (of order $1 \times m$) or all column vectors (of order $m \times 1$), for the same dimension m.

This common dimension m can be smaller than, equal to, or larger than the number n of vectors.

From now on, whenever we talk about linear combinations we will usually make the implicit assumption that all relevant vectors are of the same order.

Linear combinations: Example 1

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients* here, such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

Let $\vec{\mathbf{w}} = [4, -4],\ \vec{\mathbf{v}}_1 = [1, 2],\ \vec{\mathbf{v}}_2 = [3, 4],\ \vec{\mathbf{v}}_3 = [0, 5]$.

Then $\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 - 2\vec{\mathbf{v}}_3 = [1 + 3 - 2(0), 2 + 4 - 2(5)] = [4, -4] = \vec{\mathbf{w}}$.

Thus choosing coefficients $d_1 = 1, d_2 = 1, d_3 = -2$ gives us an expression for $\vec{\mathbf{w}}$ as in the above definition.

It follows that $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$.

Linear combinations: Example 2

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients* here, such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

Let $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{\mathbf{v}}_1 = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Then $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -0.5 \begin{bmatrix} -2 \\ -4 \end{bmatrix} = -0.5\vec{\mathbf{v}}_1 = -0.5\vec{\mathbf{v}}_1 + 0\vec{\mathbf{v}}_2$.

Thus choosing coefficients $d_1 = -0.5, d_2 = 0$ will give us an expression for $\vec{\mathbf{w}}$ as in the above definition.

It follows that $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

Linear combinations: Example 3

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients* here, such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

Let $\vec{\mathbf{w}} = [6, -4, 4],\ \vec{\mathbf{v}}_1 = [2, 0, 0],\ \vec{\mathbf{v}}_2 = [0, 8, 0],\ \vec{\mathbf{v}}_3 = [0, 0, 1]$.

Question L21.2: Is $\vec{\mathbf{w}}$ a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$?

Yes. $[6, -4, 4] = 3\,[2, 0, 0] - 0.5\,[0, 8, 0] + 4\,[0, 0, 1]$,
so that $\vec{\mathbf{w}} = 3\vec{\mathbf{v}}_1 - 0.5\vec{\mathbf{v}}_2 + 4\vec{\mathbf{v}}_3$.

Thus $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ with coefficients $d_1 = 3, d_2 = -0.5, d_3 = 4$ as in the above definition.

Linear combinations: Example 4

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients* here, such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

Let $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{\mathbf{v}}_1 = \begin{bmatrix} -5 \\ 0 \\ -6 \end{bmatrix} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$

Question L21.3: Is $\vec{\mathbf{w}}$ a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$?

No. Whenever we form a linear combination

$d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 = \begin{bmatrix} -5d_1 - 5d_2 \\ 0d_1 + 0d_2 \\ -6d_1 + 4d_2 \end{bmatrix}$ we end up with a vector that

has 0 as its second coordinate and must be different from $\vec{\mathbf{w}}$.

Linear combinations: Example 5

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \ldots, d_n$, called *coefficients* here, such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n.$

Let $\vec{\mathbf{w}} = [0, 0, 0]$, $\vec{\mathbf{v}}_1 = [1, -1, 2]$, $\vec{\mathbf{v}}_2 = [0.2, 80, 2021]$.

Question L21.4: Is $\vec{\mathbf{w}}$ a linear combination of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$?

Yes. $\vec{\mathbf{w}} = [0, 0, 0] = 0\,[1, -1, 2] + 0\,[0.2, 80, 2019] = 0\vec{\mathbf{v}}_1 + 0\vec{\mathbf{v}}_2.$

Thus $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. Choosing $d_1 = d_2 = 0$ will give us an expression for $\vec{\mathbf{w}}$ as in the above definition.

We can see from this example that a zero vector $\vec{\mathbf{0}}$ is *always* a linear combination of *any* set of vectors of the same order.

Linear combinations: Example 6

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \ldots, d_n$, called *coefficients* here, such that

$$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n.$$

Let $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{\mathbf{v}}_1 = \begin{bmatrix} -5 \\ 0 \\ -6 \end{bmatrix} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix} \quad \vec{\mathbf{v}}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

Question: Is $\vec{\mathbf{w}}$ a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$?

This is a tougher problem. Here it is difficult to *guess* values of the coefficients d_1, d_2, d_3 that would work, or to see any reason why such coefficients would not exist.

We will need a systematic method for solving such problems.

Take-home message

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \ldots, d_n$, called *coefficients* here, such that $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

The vectors $\vec{\mathbf{w}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ are assumed to be all of the same order.

The zero vector $\vec{\mathbf{w}} = \vec{\mathbf{0}}$ of the same order as $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ is always a linear combination of these vectors.