Lecture 22: Finding Linear Combinations and the Linear Span of a Set of Vectors

Winfried Just
Department of Mathematics, Ohio University

MATH3200: Applied Linear Algebra

Review: Linear combinations

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \ldots, d_n$, called *coefficients,* such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

When we talk about linear combinations, we always assume, at least implicitly, that all vectors involved have the same order.

That is, we assume that either they are all row vectors of order $1 \times m$ or all column vectors of order $m \times 1$ for the same m.

Review: Example 6 of Lecture 21

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients,* such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

Let $\quad \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{\mathbf{v}}_1 = \begin{bmatrix} -5 \\ 0 \\ -6 \end{bmatrix} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix} \quad \vec{\mathbf{v}}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

Question: Is $\vec{\mathbf{w}}$ a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$?

This is a tougher problem than for the other examples of Lecture 21. Here it is difficult to *guess* values of the coefficients d_1, d_2, d_3 that would work, or to see any reason why such coefficients would not exist.

We need a systematic method for solving such problems.

We need to solve a system of linear equations

We want to find scalars d_1, d_2, d_3 such that

$d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + d_3\vec{\mathbf{v}}_3 = \vec{\mathbf{w}}$.

We can write this as:

$$d_1 \begin{bmatrix} -5 \\ 0 \\ -6 \end{bmatrix} + d_2 \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix} + d_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -5d_1 + 5d_2 + 7d_3 \\ 0d_1 + 10d_2 + 8d_3 \\ -6d_1 + 4d_2 + 9d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

or, equivalently, as:

$$\begin{array}{rcrcrcr} -5d_1 & + & 5d_2 & + & 7d_3 & = & 1 \\ & & 10d_2 & + & 8d_3 & = & 2 \\ -6d_1 & + & 4d_2 & + & 9d_3 & = & 3 \end{array}$$

We end up with a system of linear equations with variables d_1, d_2, d_3.

We can solve this system by performing a Gaussian elimination on its augmented matrix.

Gaussian elimination for this system

$$\begin{bmatrix} -5 & 5 & 7 & 1 \\ 0 & 10 & 8 & 2 \\ -6 & 4 & 9 & 3 \end{bmatrix} \xrightarrow{R1 \mapsto R1/(-5)} \begin{bmatrix} 1 & -1 & -1.4 & -0.2 \\ 0 & 10 & 8 & 2 \\ -6 & 4 & 9 & 3 \end{bmatrix} \xrightarrow{R3 \mapsto R3+6R1}$$

$$\begin{bmatrix} 1 & -1 & -1.4 & -0.2 \\ 0 & 10 & 8 & 2 \\ 0 & -2 & 0.6 & 1.8 \end{bmatrix} \xrightarrow{R2 \mapsto R1/10} \begin{bmatrix} 1 & -1 & -1.4 & -0.2 \\ 0 & 1 & 0.8 & 0.2 \\ 0 & -2 & 0.6 & 1.8 \end{bmatrix}$$

$$\xrightarrow{R3 \mapsto R3+2R2} \begin{bmatrix} 1 & -1 & -1.4 & -0.2 \\ 0 & 1 & 0.8 & 0.2 \\ 0 & 0 & 2.2 & 2.2 \end{bmatrix} \xrightarrow{R3 \mapsto R3/2.2} \begin{bmatrix} 1 & -1 & -1.4 & -0.2 \\ 0 & 1 & 0.8 & 0.2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The resulting row echelon form is extended matrix of the equivalent system

$$\begin{array}{rcrcrcr} d_1 & - & d_2 & - & 1.4d_3 & = & -0.2 \\ & & d_2 & + & 0.8d_3 & = & 0.2 \\ & & & & d_3 & = & 1 \end{array}$$

Finding the coefficients of the linear combination

We found that any scalars d_1, d_2, d_3 such that

$d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + d_3\vec{\mathbf{v}}_3 = \vec{\mathbf{w}}$

form the solutions the system of linear equations

$$\begin{array}{rcrcrcr} d_1 & - & d_2 & - & 1.4d_3 & = & -0.2 \\ & & d_2 & + & 0.8d_3 & = & 0.2 \\ & & & & d_3 & = & 1 \end{array}$$

Question L22.1: Based on this information, what can you say about these linear combinations?

The above system has a unique solution. This solution gives us coefficients $d_1 = 0.6, d_2 = -0.6$, and $d_3 = 1$. We can then conclude that

$\vec{\mathbf{w}} = 0.6\vec{\mathbf{v}}_1 - 0.6\vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3$

is indeed a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$.

We can also conclude that there is no other choice for the coefficients.

A general method for finding coefficients of linear combinations

Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n, \vec{\mathbf{w}}$ be given vectors of the same order.
The following method allows you to determine whether $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ and find coefficients if it is.

1. Form a matrix $\mathbf{A}$ that either has $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ (if they are column vectors) or $\vec{\mathbf{v}}_1^T, \vec{\mathbf{v}}_2^T, \dots, \vec{\mathbf{v}}_n^T$ as its successive columns.
2. Let $\vec{\mathbf{b}}$ be either $\vec{\mathbf{w}}$ or $\vec{\mathbf{w}}^T$, depending on whether $\vec{\mathbf{w}}$ is a column or a row vector.
3. Solve the linear system with extended matrix $[\mathbf{A}, \vec{\mathbf{b}}]$.
4. If this system is *inconsistent,* $\vec{\mathbf{w}}$ is *not* a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$.
5. If this system is *consistent,* $\vec{\mathbf{w}}$ *is* a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ and *each* solution vector will give you coefficients of a linear combination.

Note that we have not specified here whether the variables of the linear system should be called $d_1, \dots, d_n$ or $x_1, \dots, x_n$. This does not matter.

An example with row vectors

Let $\vec{\mathbf{w}} = [1, 2], \quad \vec{\mathbf{v}}_1 = [1, 3], \quad \vec{\mathbf{v}}_2 = [1, -1]$.

We want to find d_1, d_2 such that $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2$.

Question L22.2: Can we find all such vectors of possible coefficients d_1 and d_2 by solving the following system of linear equations?

$$\begin{array}{rcrcl} d_1 & + & 3d_2 & = & 1 \\ d_1 & - & d_2 & = & 2 \end{array}$$

No. This system has a unique solution with $d_1 = 1.75$ and $d_2 = -0.25$. However,

$1.75\vec{\mathbf{v}}_1 - 0.25\vec{\mathbf{v}}_2 = 1.75\,[1, 3] - 0.25\,[1, -1] = [1.5, 5.5] \neq \vec{\mathbf{w}}$.

Question L22.3: What went wrong here?

We need to change row vectors to column vectors

Recall that we let $\vec{\mathbf{w}} = [1, 2]$, $\vec{\mathbf{v}}_1 = [1, 3]$, $\vec{\mathbf{v}}_2 = [1, -1]$
and that we want to find d_1, d_2 such that $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2$.

This means: $[1, 2] = d_1[1, 3] + d_2[1, -1] = [d_1 + d_2, 3d_1 - d_2]$.

So, instead of solving the system given at the previous slide,

we must solve the system
$$\begin{array}{rcrcl} d_1 & + & d_2 & = & 1 \\ 3d_1 & - & d_2 & = & 2 \end{array}$$

where the *transposes* of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are the *columns* of the

coefficient matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

This corrected system has a unique solution with $d_1 = 0.75$ and $d_2 = 0.25$. These coefficients work out right:

$0.75\vec{\mathbf{v}}_1 + 0.25\vec{\mathbf{v}}_2 = 0.75\,[1, 3] + 0.25\,[1, -1] = [1, 2] = \vec{\mathbf{w}}$.

Systems of linear equations as linear combinations

Consider a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Let $\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{a}}_n$ be the column vectors of the coefficient matrix $\mathbf{A}$, and let $\vec{\mathbf{b}}$ be the column vector that represents the right-hand side:

$$\vec{\mathbf{a}}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \vec{\mathbf{a}}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \quad \dots \quad \vec{\mathbf{a}}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \quad \vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Then the above system can be written as

$x_1\vec{\mathbf{a}}_1 + x_2\vec{\mathbf{a}}_2 + \cdots + x_n\vec{\mathbf{a}}_n = \vec{\mathbf{b}}$.

An observation: Consistency of systems and linear combinations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Let $\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{a}}_n$ be the column vectors of the coefficient matrix $\mathbf{A}$, and let $\vec{\mathbf{b}}$ be the column vector that represents the right-hand side.

By writing the system as $x_1\vec{\mathbf{a}}_1 + x_2\vec{\mathbf{a}}_2 + \cdots + x_n\vec{\mathbf{a}}_n = \vec{\mathbf{b}}$ we essentially get a one-line proof of the following result:

Theorem

In the notation introduced above, a system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of linear equations is consistent if, and only if, the vector $\vec{\mathbf{b}}$ is a linear combination of the columns $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n$ of the coefficient matrix $\mathbf{A}$.

The linear span

Definition

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ be a set of vectors of the same order. The *linear span* of these vectors is the set

$span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$

of all linear combinations of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$.

At some level, this definition just says that the phrases "$\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$" and "$\vec{\mathbf{w}}$ is in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$" mean exactly the same thing.

But the linear span is usually a very large set of vectors with very interesting properties, as we will see soon. Therefore it is useful to have a standard notation and standard name for this set.

An observation: Consistency of systems and the linear span

Two slides earlier we saw a version of the following theorem:

Theorem

A system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of linear equations is consistent if, and only if, the vector $\vec{\mathbf{b}}$ is a linear combination of the columns $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n$ of the coefficient matrix $\mathbf{A}$.

We can rephrase this theorem in the language of linear spans as follows:

Theorem

A system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of linear equations is consistent if, and only if, the vector $\vec{\mathbf{b}}$ is the linear span $span(\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n)$ of the columns $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n$ of the coefficient matrix $\mathbf{A}$.

The linear span: Some examples

On this slide, let $\mathbb{R}^3$ denote the set of all 1×3 row vectors. Then:

span$([1, -1, 2])$ is the set of all vectors of the form $d[1, -1, 2] = [d, -d, 2d]$. This set is a *line* in $\mathbb{R}^3$.

span$([1, 0, 0], [0, 1, 0], [0, 0, 1])$ is $\mathbb{R}^3$ itself, as for any given x, y, z we can write $[x, y, z] = x[1, 0, 0] + y[0, 1, 0] + z[0, 0, 1]$.

Question L22.4: What is *span*$([1, 0, 0], [0, 1, 0])$?

The x-y plane.

Question: What is *span*$([1, 2, 4], [5, -1, 7], [3, 8, 5])$?

This is a tough one. We can determine for every *individual* vector in $\mathbb{R}^3$ whether it is in the linear span by using the method you learned earlier in this lecture and solving a corresponding system of linear equations. But finding *all* vectors in this linear span will require new methods.

Take-home message

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients,* such that
$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

The vectors $\vec{\mathbf{w}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ are assumed to be all of the same order.

The set of all linear combinations of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ is denoted by $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ and called the *the linear span* of these vectors.

Determining whether a given vector is in the linear span of a given set of vectors and finding coefficients for linear combinations boils down to solving a system of linear equations. A detailed procedure is outlined on slide 7.

A linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent if, and only if, $\vec{\mathbf{b}}$ is in the linear span of the column vectors of its coefficient matrix $\mathbf{A}$.