# Lecture 23: Vector Spaces

Winfried Just
Department of Mathematics, Ohio University

MATH3200: Applied Linear Algebra

# Review: The linear span and $\mathbb{R}^n$

## Definition

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ be a set of vectors of the same order. The *linear span* of these vectors is the set

$span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n)$ that consists

of all linear combinations of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$.

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be a set of row or column vectors of dimension $n$.

We treat these vectors as points in $\mathbb{R}^n$, where $\mathbb{R}$ denotes the real line and $\mathbb{R}^n$ denotes the Euclidean space of dimension $n$.

# How should we imagine the linear span?

Let $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ be the linear span of some vectors in $\mathbb{R}^n$ for some positive integers $k, n$.

**What kind of subsets of $\mathbb{R}^n$ are of this form?**

**How can we imagine linear spans geometrically?**

In Lecture 22 we saw an example where the linear span is a line,
an example where the linear span is a plane,
and an example where it is a three-dimensional space. Recall that:

- $span([1, -1, 2])$ is a line.
- $span([1, 0, 0], [0, 1, 0])$ is the $x$-$y$ plane.
- $span([1, 0, 0], [0, 1, 0], [0, 0, 1])$ is $\mathbb{R}^3$ itself.

Recall Question 43.5 of Module 43:

**Question L23.1:** Is every line in $\mathbb{R}^3$ a linear span of some vectors?

# The zero vector is in every linear span

**No.**

Consider, for example, the line $L$ that consists of all vectors $[x, 5, 0]$. This line doesn't go through the origin $\vec{\mathbf{0}} = [0, 0, 0]$.

But for any set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ of the same dimension, we get the zero vector

$\vec{\mathbf{0}} = 0\vec{\mathbf{v}}_1 + 0\vec{\mathbf{v}}_2 + \cdots + 0\vec{\mathbf{v}}_n$

as a linear combination of these vectors.

So, $\vec{\mathbf{0}}$ must be in the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$.

Since $L$ doesn't go through the origin, it cannot be a linear span.

Similarly, a plane can be a linear span only if it goes through the origin.

# Linear subspaces of $\mathbb{R}^3$

Let $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$ be the linear span of some vectors in $\mathbb{R}^3$.

Then $V$ is a subset of $\mathbb{R}^3$ called a *linear subspace* of $\mathbb{R}^3$.

We found the following kinds of linear subspaces of $\mathbb{R}^3$:

- Lines through the origin.
- Planes that contain the origin.
- $\mathbb{R}^3$ itself.

**Question L23.2:** Did we miss any other possiblity?

**Yes.** We need to add to our list:

- $\{[0,0,0]\} = \{\vec{\mathbf{0}}\} = span(\vec{\mathbf{0}})$.

Now our list is complete and describes all possibilities.

# Review: Properties of the linear span

Recall the following results that were proved in Module 43 and Conversation 24:

**Proposition**

*Let $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\}$ be a subset of $\mathbb{R}^n$, let $\lambda$ be a scalar (a number in $\mathbb{R}$), and let $\vec{\mathbf{w}}$ be in $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$.*

*Then $\lambda\vec{\mathbf{w}}$ is also in $V$.*

*Thus $V$ is closed under multiplication by scalars.*

**Proposition**

*Let $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\}$ be a subset of $\mathbb{R}^n$, and let $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ be in $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$.*

*Then $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is also in $V$.*

*Thus $V$ is closed under addition of vectors.*

# Linear subspaces of $\mathbb{R}^n$ and vector spaces

**Theorem**

*Let $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\}$ be a subset of $\mathbb{R}^n$, let $d_1, d_2, \ldots, d_\ell$ be scalars, and let $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \ldots, \vec{\mathbf{w}}_\ell$ be in $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$.*

*Then $d_1\vec{\mathbf{w}}_1 + d_2\vec{\mathbf{w}}_2 + \cdots + d_\ell\vec{\mathbf{w}}_\ell$ is also in $V$.*

*Thus $V$ is closed under linear combinations.*

We call a subset $V$ of some $\mathbb{R}^n$ that is of the form $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$ for some subset $S = \{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\}$ of $\mathbb{R}^n$ a *linear subspace of $\mathbb{R}^n$.*

The set $S$ is then called *a spanning set of $V$* and we will sometimes write $V = span(S)$ instead of $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$.

The theorem tells us that within $V$ the operations of addition of vectors, multiplication by scalars, and formation of linear combinations can always be performed. They also will have analogous properties as in $\mathbb{R}^n$ itself, such as distributivity and commutativity laws $\lambda(\vec{\mathbf{u}} + \vec{\mathbf{w}}) = \lambda\vec{\mathbf{u}} + \lambda\vec{\mathbf{w}}$ and $\vec{\mathbf{u}} + \vec{\mathbf{w}} = \vec{\mathbf{w}} + \vec{\mathbf{u}}$.

# Linear subspaces of $\mathbb{R}^n$ *vs.* vector spaces

**Theorem**

*Let $n$ be a positive integer. A subset $V$ of $\mathbb{R}^n$ is of the form $V = span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ for some vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ if, and only if, it is closed under multiplication by scalars and addition of vectors, that is, for every scalar $\lambda$ and $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ in $V$ the vectors $\lambda\vec{\mathbf{w}}$ and $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ are also in $V$.*

We have proved the *only if* direction of the theorem.

A proof of the *if* direction will be omitted here.

In general, in an abstract *vector space* the vectors can be any mathematical objects for which addition and multiplication by scalars is defined and obeys certain properties. The theorem tells us that if the vectors are our usual points in $\mathbb{R}^n$ with the usual operations, then a vector space is the same thing as a linear subspace of $\mathbb{R}^n$.

**For the purpose of this course, the phrase vector space will always refer to a linear subspace of $\mathbb{R}^n$ for some positive integer $n$.**

# Examples of vector spaces not covered in this course

In general, scalars can also be complex numbers, restricted to rational numbers, or even fancy mathematical objects that belong to so-called *algebraic fields.*

- The set of all $m \times 1$ matrices whose entries are *complex numbers* is an example of a vector space over the algebraic field $\mathbb{C}$ of all complex numbers.
- The set of all functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space.
- The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space.
- The set of all polynomials with real coefficients is a vector space over the algebraic field $\mathbb{R}$.
- The set of all polynomials with complex coefficients is a vector space over the algebraic field $\mathbb{C}$.

# Take-home message

We call a subset $V$ of some $\mathbb{R}^n$ that is of the form $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$ for some subset $S = \{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\}$ of $\mathbb{R}^n$ a *linear subspace of $\mathbb{R}^n$.*

The set $S$ is then called *a spanning set of $V$* and we will sometimes write $V = span(S)$ instead of $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$.

Each linear subspace of $\mathbb{R}^3$ is of one of the following forms:

- The set $\{\vec{\mathbf{0}}\}$ that consists only of the origin.
- A line through the origin.
- A plane that contains the origin.
- $\mathbb{R}^3$ itself.

*In this course,* the phrase *vector space* will always mean "a linear subspace of $\mathbb{R}^n$ for some positive integer $n$."