Lecture 24: More on Linear Dependence and Linear Independence

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MATH 3200: Applied Linear Algebra

Review: Linear (in)dependence, tentative definition

Recall the following definition from Conversation 25:

Definition (Tentative)

Consider a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ of vectors of the same order.

This set is *linearly dependent* if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.

This set is *linearly independent* if, and only if, it is not linearly dependent.

This definition is more intuitive than the official one when we have a set of at least two vectors, so that $k > 1$.

For a set $S = \{\vec{\mathbf{v}}_1\}$ of $k = 1$ vector the definition still works, but it is less intuitive. It requires that $\vec{\mathbf{v}}_1$ is not a linear combination of the *other* vectors in the set.

When $k = 1$, there are no other vectors in the set S, so the set of vectors in S other than $\vec{\mathbf{v}}_1$ would be *the empty set*, denoted by $\emptyset$, the set that has no elements whatsoever.

Linear (in)dependence of a set of one vector

The empty set $\emptyset$ is a bit of an oddball and doesn't fit our usual definitions. To make things work out nicely, we will need to arbitrarily define that

$span(\emptyset) = \{\vec{\mathbf{0}}\}$.

This can be done *only* in a situation where we know what the order of the vectors is that we are interested in, which will depend on the context.

Now consider a set $S = \{\vec{\mathbf{v}}_1\}$ of $k = 1$ vector.

When $\vec{\mathbf{v}}_1 \neq \vec{\mathbf{0}}$, then $\vec{\mathbf{v}}_1$ is not in $span(\emptyset)$, which means that $\vec{\mathbf{v}}_1$ is not a linear combination of the other vectors in this set, and the set S will be linearly independent.

When $\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$, then $\vec{\mathbf{v}}_1$ would be in $span(\emptyset)$, that is, would be considered a linear combination of the vectors in the empty set, and the set S will be linearly dependent.

Thus our tentative definition works exactly as the official definition also in the case when the set S contains exactly $k = 1$ vector.

Linear (in)dependence of larger sets

Suppose $S^- = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ and $S^+ = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}, \dots, \vec{\mathbf{v}}_\ell\}$ are sets of vectors that are all of the same order such that S^- is a subset of S^+, which means that every vector in S^- is also in S^+.

Question L24.1: Suppose S^- is linearly dependent.
Can we conclude that S^+ is also linearly dependent?

Question L24.2: Suppose S^- is linearly independent.
Can we conclude that S^+ is also linearly independent?

Answer L24.1: Yes. If S^- is linearly dependend, then some vector $\vec{\mathbf{v}}_i$ is a linear combination of the other vectors in S^-. But then $\vec{\mathbf{v}}_i$ is also in S^+ and is a linear combination of the other vectors in the set S^+.

Answer L24.2: No. Here S^+ could be linearly independend, but does not need to be. For example, if $\vec{\mathbf{v}}_{k+1}$ is a linear combination of the vectors in S^-, then S^+ will be linearly dependent.

Linear (in)dependence of subsets and supersets

Theorem

Suppose $S^- = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ *and* $S^+ = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}, \dots, \vec{\mathbf{v}}_\ell\}$ *are sets of vectors that are all of the same order such that every vector in* S^- *is also in* S^+. *This can also be expressed by writing that* S^- *is a subset of* S^+ *and* S^+ *is a superset of* S^-. *Then*

- *If* S^- *is linearly dependent, then* S^+ *is also linearly dependent.*
- *If* S^+ *is linearly independent, then* S^- *is also linearly independent.*

A sketch of the proof of the first item was already given in the answer to Question L24.1.

The second item follows from the first: Assume S^+ is linearly independent. If S^- were linearly dependent, then S^+ would also need to be linearly dependent, which leads to a contradiction and thus cannot be the case.

An advantage of the official definition

Definition (Official)

A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of vectors of the same order is *linearly dependent* if, and only if, there are scalars $c_1, c_2, \ldots, c_k$, not all of them zero, so that

$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

This set is *linearly independent* if, and only if, it is not linearly dependent.

Suppose $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is a given set of vectors and we want to know whether this set is linearly dependent or linearly independent. The official definition allows us to reduce this problem to finding all vectors $\vec{\mathbf{c}} = [c_1, c_2, \ldots, c_k]^T$ of coefficients such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

When the only vector of such coefficients is $\vec{\mathbf{c}} = [0, 0, \ldots, 0]^T = \vec{\mathbf{0}}$, then the set S is linearly independent; otherwise it is linearly dependent.

Determining linear (in)dependence: Example 1

Let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$, where $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In order to determine whether S is linearly dependent or linearly independent, we need to find all vectors of coefficients c_1, c_2, c_3 such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$.

As we learned in Lecture 22 and Module 42, this boils down to finding all solutions of the following system of linear equations:

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & c_3 & = & 0 \\ & & c_2 & + & c_3 & = & 0 \\ 3c_1 & + & 5c_2 & + & c_3 & = & 0 \end{array}$$

We can solve this system by first performing Gaussian elimination on the extended matrix and then using back-substitution.

Example 1 continued: Gaussian elimination

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 5 & 1 & 0 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 3R1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \mapsto R3 + R2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{R3 \mapsto -R3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The resulting matrix in row-echelon form is the extended matrix of the following equivalent system:

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & + & c_3 & = & 0 \\ & & c_2 & + & c_3 & = & 0 \\ & & & & c_3 & = & 0 \end{array}$$

Question L24.3: Is the set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ linearly dependent or linearly independent?

Linearly independent since the only solution of this linear system gives us coefficients $c_1 = c_2 = c_3 = 0$.

Determining linear (in)dependence: Example 2

Let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$, where

$\vec{\mathbf{v}}_1 = [1, 0, 3]$, $\vec{\mathbf{v}}_2 = [2, 2, 4]$, $\vec{\mathbf{v}}_3 = [1, 2, 1]$

In order to determine whether S is lineary dependent or linearly independent, we need to find all vectors of coefficients c_1, c_2, c_3 such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$.

As we learned in Lecture 22 and Module 42, this boils down to finding all solutions of the following system of linear equations, where the vectors $\vec{\mathbf{v}}_1^T, \vec{\mathbf{v}}_2^T, \vec{\mathbf{v}}_3^T$ are the columns of the oefficient matrix:

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & + & c_3 & = & 0 \\ & & 2c_2 & + & 2c_3 & = & 0 \\ 3c_1 & + & 4c_2 & + & c_3 & = & 0 \end{array}$$

We can solve this system by first performing Gaussian elimination on the extended matrix and then using back-substitution.

Example 2 continued: Gaussian elimination

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 3 & 4 & 1 & 0 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 3R1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \mapsto R3 + R2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R3/2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The resulting matrix in row-echelon form is the extended matrix of the following equivalent system:

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & + & c_3 & = & 0 \\ & & c_2 & + & c_3 & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

Question L24.4: Is the set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ linearly dependent or linearly independent?

Linearly dependent since this linear system is underdetermined and has nonzero solutions, for example $c_1 = 1, c_2 = -1, c_3 = 1$.

A general observation

Let us make an interesting observation about this procedure.
In essence, it gives a proof of the following theorem:

Theorem

Consider a homogenous system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *and let* $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n$ *bet the columns of the coefficient matrix. Then*

- *The set* $\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n\}$ *is linearly dependent if, and only if, the system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *is underdetermined.*
- *The set* $\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n\}$ *is linearly independent if, and only if, the system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has exactly one solution.*

Note that since a homogenous system of linear equations always is consistent, the two items of this theorem make the exact same assertion, but in two different ways.

Another characterization of linear independence

Theorem

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be a set of vectors of the same order. Then these vectors are linearly independent if, and only if, every vector $\vec{\mathbf{w}}$ in span$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ can be expressed as

$$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$$

for exactly one choice of the coefficients $c_1, c_2, \ldots, c_k$.

Before we prove this theorem, let us restate it in another way:

Theorem

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be a set of vectors of the same order. Then these vectors are linearly dependent if, and only if, some vector $\vec{\mathbf{w}}$ in span$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ can be expressed as

$$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_k\vec{\mathbf{v}}_k$$

for at least two different choices of the coefficients $c_1, c_2, \ldots, c_k$ and $d_1, d_2, \ldots, d_k$.

Proof of the second version of the theorem, one direction

Theorem

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ be a set of vectors of the same order. Then these vectors are linearly dependent if, and only if, some vector $\vec{\mathbf{w}}$ in span$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$ can be expressed as

$$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_k\vec{\mathbf{v}}_k$$

for at least two different choices of the coefficients $c_1, c_2, \dots, c_k$ and $d_1, d_2, \dots, d_k$.

If the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is linearly dependent, then according to the official definition there are coefficients $c_1, c_2, \dots, c_k$, not all of them zero, so that $\vec{\mathbf{0}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$.

On the other hand, we always have $\vec{\mathbf{0}} = 0\,\vec{\mathbf{v}}_1 + 0\,\vec{\mathbf{v}}_2 + \cdots + 0\,\vec{\mathbf{v}}_k$.

Since the vector $\vec{\mathbf{0}}$ is always in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$, with taking $d_1 = d_2 = \cdots = d_k = 0$ and $\vec{\mathbf{w}} = \vec{\mathbf{0}}$, we have *a different choice* of coefficients for expressing *some* vector $\vec{\mathbf{w}}$ in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$ as a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k$.

Proof of the second version of the theorem, other direction

Theorem

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ *be a set of vectors of the same order. Then these vectors are linearly dependent if, and only if, some vector* $\vec{\mathbf{w}}$ *in span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ *can be expressed as*

$$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_k\vec{\mathbf{v}}_k$$

for at least two different choices of the coefficients $c_1, c_2, \ldots, c_k$ *and* $d_1, d_2, \ldots, d_k$.

Assume *some* vector $\vec{\mathbf{w}}$ in *span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ can be expressed as
$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_k\vec{\mathbf{v}}_k$, where
$c_i \neq d_i$ *for some* i. Then

$$\vec{\mathbf{w}} - \vec{\mathbf{w}} = (c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k) - (d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_k\vec{\mathbf{v}}_k),$$

$$\vec{\mathbf{0}} = (c_1 - d_1)\vec{\mathbf{v}}_1 + \cdots + (c_i - d_i)\vec{\mathbf{v}}_i + \cdots + (c_k - d_k)\vec{\mathbf{v}}_k,$$

and since $c_i - d_i \neq 0$, the set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is linearly dependent according to the official definition.

Take-home message

When we define the linear span of the *empty set* of vectors as $span(\emptyset) = \{\vec{\mathbf{0}}\}$, then we can extend the tentative definition of linear (in)dependence that was given in Conversation 25 to the case sets $\{\vec{\mathbf{v}}_1\}$ of exactly one vector. I will then still be equivalent to the official definition that was also given in Conversation 25.

A set $\{\vec{\mathbf{v}}_1\}$ is linearly independent if, and only if, $\vec{\mathbf{v}}_1 \neq \vec{\mathbf{0}}$.

Suppose $S^- = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ and $S^+ = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}, \dots, \vec{\mathbf{v}}_\ell\}$ are sets of vectors that are all of the same order such that every vector in S^- is also in S^+. This can also be expressed by writing that S^- is a *subset* of S^+ and S^+ is a *superset* of S^-. Then

- If S^- is linearly dependent, then S^+ is also linearly dependent.
- If S^+ is linearly independent, then S^- is also linearly independent.

Take-home message, continued

In order to determine whether a given set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots \vec{\mathbf{v}}_k\}$ is linearly dependent or linearly independent, we need to find the set of all possible coefficients for the linear combinations $\vec{\mathbf{0}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$ by following the procedure that we learned in Lecture 22 and Module 42, which boils down to solving a homogenous system of linear equations.

When the resulting homogenous system of linear equations is underdetermined, there are choices of the coefficients that are not all equal to zero, and the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots \vec{\mathbf{v}}_k\}$ is linearly dependent. On the other hand, when the zero vector is the only solution of this system of linear equations, the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots \vec{\mathbf{v}}_k\}$ is linearly independent.

A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is linearly independent if, and only if, *every* vector $\vec{\mathbf{w}}$ in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$ can be expressed as

$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$

for exactly one choice of the coefficients $c_1, c_2, \dots, c_k$.