Lecture 25A: Definitions of Bases

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MATH 3200: Applied Linear Algebra

Review: Bases of vector spaces

In Conversation 26 a basis of a vector space was defined as follows:

Definition

Let V be a vector space. A linearly independent spanning set of V is called a *basis* of V.

We then saw that we can start with any spanning set S of V and successively remove vectors from S until we end up with a linearly independent set that has the same linear span and must be a basis of V. This proves the following result:

Theorem

Let $V = span(S)$ for some set of vectors S. Then S contains a subset B that is a basis of V.

Minimal spanning sets

The linearly independent subset that we obtain by this removal procedure is a *minimal* spanning set for V in the sense that no subset of it that contains fewer vectors can be a spanning set of V.

To see this, notice that if we remove a vector $\vec{\mathbf{v}}$ from a linearly independent set S, then $\vec{\mathbf{v}}$ is not in $span(S^-)$ for the resulting subset S^- of S, since by our tentative definition of linear independence $\vec{\mathbf{v}}$ is not in the linear span of the other vectors in S^-.

But $\vec{\mathbf{v}}$ is in $span(S)$, so whenever we form a smaller subset S^- of S we lose some vectors from the linear span of S.

Some textbooks define a basis of a vector space V as a minimal spanning set of V; we can see from the above discussion that this definition is equivalent to ours.

Review: The dimension of a vector space V

We saw in Conversation 26 that there are usually many different bases for a given vector space V, but all of them have the same size, which is the *dimension* of V.

Theorem

Let V be any vector space and let B_1, B_2 be two bases of V. Then B_1 and B_2 have the same size.

Definition

Let V be any vector space. Then the *dimension* of V, denoted by $dim(V)$, is the size of any basis of V.

An oddball: The space $V = \{\vec{\mathbf{0}}\}$

Let us now consider $V = \{\vec{\mathbf{0}}\} = span(\vec{\mathbf{0}})$.

Question L25.1 What basis for V do you obtain from our removal procedure when you start with the spanning set $\{\vec{\mathbf{0}}\}$ for V?

Since the set $\{\vec{\mathbf{0}}\}$ is linearly dependent, we must remove $\vec{\mathbf{0}}$ and end up with the empty set as our basis.

This set has zero elements and we conclude that $dim(\{\vec{\mathbf{0}}\}) = 0$, which nicely conforms to our geometric intuition.

It also follows that we need to treat the empty set as a linearly independent set, which is less intuitive. Generally speaking, the vector space $V = \{\vec{\mathbf{0}}\}$ is an oddball. It is the only vector space among those studied in this course that does not have infinitely many elements, infinitely many spanning sets, or infinitely many bases. It is not a very useful vector space all by itself, but we need to know about it as it pops up in some calculations.

Another example of a spanning set

Let $S = \{[1,0],[0,1],[1,1]\}$ and $V = span(S)$.

Since $[1,0] = [1,1] - [0,1]$, we can remove $[1,0]$ from S and obtain a basis $B_1 = \{[0,1],[1,1]\}$ of V.

Since $[0,1] = [1,1] - [1,0]$, we can remove $[0,1]$ from S and obtain another basis $B_2 = \{[1,0],[1,1]\}$ of V.

Since $[1,1] = [1,0] + [0,1]$, we can remove $[1,1]$ from S and obtain yet another basis $B_3 = \{[1,0],[0,1]\}$ of V.

Each of these bases has size 2, so that $dim(V) = 2$ and $V = \mathbb{R}^2$.

Alternatively, we could start with the empty set $\emptyset$, then add $[1,0]$, which is not in $span(\emptyset)$, and finally add $[0,1]$, which is not in $span([1,0])$. This will give us the linearly independent set B_3.

This set is a *maximal* linearly independent subset of V in the sense that we could not add more vectors from V without creating a linearly dependent set.

Yet another definition of a basis

Let $S = \{[1,0],[0,1],[1,1]\}$ and $V = span(S)$.

Smilarly, we could start with the empty set $\emptyset$, then add $[0,1]$, which is not in $span(\emptyset)$, and finally add $[1,1]$, which is not in $span([0,1])$. This will give us the linearly independent set $B_1 = \{[0,1],[1,1]\}$, which is also a *maximal* linearly independent subset of V in the sense that we could not add more vectors from V without creating a linearly dependent set.

We could also produce the set B_2 of the previous slide in this way.

Some textbooks define a basis of a vector space V as a maximal linearly independent subset of V. The examples on this slide and the previous one illustrate why this definition is equivalent to ours.

Take-home message: Bases of vector spaces

Let V be a vector space. A set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ such that $V = span(S) = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ is called a *spanning set* of V.

A linearly independent spanning set of V is called a *basis* of V.

A basis of V can also be defined as a *minimal* spanning set of V, that is, a spanning set S of V such that no smaller subset of S remains a spanning set of V.

Moreover, a basis of V can be defined as a *maximal* linearly independent subset of V, that is, a linearly independent subset B of V to which we cannot add any vectors of V without making the resulting set linearly dependent.

Every two bases for the same vector space V have the same size. This size is called the *dimension* of V and denoted by $dim(V)$.

The basis of a vector space of the form $V = \{\vec{\mathbf{0}}\}$ is the empty set and $dim(\{\vec{\mathbf{0}}\}) = 0$.