Lecture 25B: Applications of Bases Parametrization and Change of Bases

Winfried Just
Department of Mathematics, Ohio University

MATH 3200: Applied Linear Algebra

The key property of bases

Definition

Let V be a vector space. A linearly independent spanning set of V is called a *basis* of V.

While there are several ways of defining bases, our definition is the most convenient one for applications. Recall from Lecture 24:

Theorem

Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be a set of vectors of the same order. Then these vectors are linearly independent if, and only if, every vector $\vec{\mathbf{w}}$ in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ can be expressed as

$$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$$

for exactly one choice of the coefficients $c_1, c_2, \ldots, c_k$.

Thus if $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$, then every vector $\vec{\mathbf{w}}$ in $V = span(B)$ can be expressed as a linear combination of vectors in B for exactly one choice of coefficients $c_1, c_2, \ldots, c_k$. These coefficients can then be treated as the *coordinates* of $\vec{\mathbf{w}}$ with respect to B.

Review: The vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n$

For example, consider the following vectors in $\mathbb{R}^n$ for a given n:

$$\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \ldots \quad \vec{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

When we treat $\mathbb{R}^n$ as a set of row vectors, we would use the same notation $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n$ for

$$\vec{\mathbf{e}}_1 = [1, 0, \ldots, 0] \qquad \vec{\mathbf{e}}_2 = [0, 1, \ldots, 0] \quad \ldots \quad \vec{\mathbf{e}}_n = [0, 0, \ldots, 1]$$

Either way, these vectors are called the *standard basis vectors* of $\mathbb{R}^n$, and $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n\}$ forms the *standard basis* of $\mathbb{R}^n$.

In the remainder of this lecture we will work with **column vectors.**

Cartesian coordinates

Consider any vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ for a given n. Then

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

In the notation of the standard basis vectors, we can write this as

$\vec{\mathbf{x}} = x_1\vec{\mathbf{e}}_1 + x_2\vec{\mathbf{e}}_2 + \cdots + x_n\vec{\mathbf{e}}_n,$

and the coefficients of this linear combination are unique.

So we can uniquely identify each vector in $\mathbb{R}^n$ in terms of

these coefficients $\quad \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

which are its **Cartesian coordinates**.

(Alternative) coordinates with respect to a given basis

Now let $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be any basis for a vector space V.

Then any vector $\vec{\mathbf{w}}$ in V can be expressed as a linear combination
$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$,
and the coefficients of this linear combination are unique.

So we can uniquely identify each vector in V in terms of

these coefficients $\vec{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$

which are its **(alternative) coordinates** with respect to B.

We will use the word "alternative" when B is *not* the standard basis.

We can also call $\vec{\mathbf{c}}$ a *parametrization* of $\vec{\mathbf{w}}$ with respect to B.

What are parametrizations good for?

Why would anyone ever want to use alternative coordinates?

Let $\vec{\mathbf{v}}_1 = [1, 2, 3, 4, 5, 6, 7, 8, 9]^T, \vec{\mathbf{v}}_2 = [9, 8, 7, 6, 5, 4, 3, 2, 1]^T$, let $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$, and let $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$.

Then B is a basis for V, and V is a 2-dimensional subspace of $\mathbb{R}^9$.

Let $\vec{\mathbf{w}} = [-8, -6, -4, -2, 0, 2, 4, 6, 8]^T = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$.

Then $\vec{\mathbf{w}}$ is in V, and the **alternative coordinates** of $\vec{\mathbf{w}}$ with respect to B are $\vec{\mathbf{c}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Note that we need only 2 numbers for representing $\vec{\mathbf{w}}$ in alternative coordinates, but we would need 9 numbers to represent $\vec{\mathbf{w}}$ in Cartesian coordinates.

Situations like this one occur frequently when we want to represent vectors in a low-dimensional subspace of a high-dimensional space. Think about how much computer memory could be saved if we would work here with vectors in, for example $\mathbb{R}^{10,000}$ instead of $\mathbb{R}^9$.

Switching between Cartesian and alternative coordinates: Some easy examples

Let $\vec{\mathbf{v}}_1 = [1, 2, 3, 4, 5, 6, 7, 8, 9]^T, \vec{\mathbf{v}}_2 = [9, 8, 7, 6, 5, 4, 3, 2, 1]^T$, let $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$, and let $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$.

Let $\vec{\mathbf{w}} = [10, 10, 10, 10, 10, 10, 10, 10, 10]^T$.

Question L25.2: Find the **alternative coordinates** of $\vec{\mathbf{w}}$ with respect to B.

Here $\vec{\mathbf{w}} = \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$.

The **alternative coordinates** of $\vec{\mathbf{w}}$ with respect to B are $\vec{\mathbf{c}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Question L25.3: Find the **Cartesian coordinates** of the vector $\vec{\mathbf{w}}$ that has **alternative coordinates** $\vec{\mathbf{c}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Here $\vec{\mathbf{w}} = 2\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$. Thus the **Cartesian coordinates** $\vec{\mathbf{x}}$ of $\vec{\mathbf{w}}$ are $\vec{\mathbf{x}} = [-7, -4, -1, 2, 5, 8, 11, 14, 17]^T$

Expressing $\vec{\mathbf{c}}$ in Cartesian coordinates: General method

Suppose $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is a basis of a vector space V and the basis vectors are written in **Cartesian coordinates.**
Consider a vector $\vec{\mathbf{w}}$ in V with **alternative coordinates** $\vec{\mathbf{c}} = [c_1, \ldots, c_k]^T$ with respect to B.

Then we can get the **Cartesian coordinates** $\vec{\mathbf{x}}$ of $\vec{\mathbf{w}}$ by computing the following linear combination:

$$\vec{\mathbf{x}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$$

For the example of Question L25.3 we got from this calculation:
$\vec{\mathbf{w}} = 2\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 = 2[1, 2, 3, 4, 5, 6, 7, 8, 9]^T - [9, 8, 7, 6, 5, 4, 3, 2, 1]^T$
$\vec{\mathbf{x}} = [-7, -4, -1, 2, 5, 8, 11, 14, 17]^T$

When we write the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$ in the given order as the columns of a matrix $\mathbf{B}$, then the above expression for $\vec{\mathbf{x}}$ can be written in matrix notation as:

$$\vec{\mathbf{x}} = \mathbf{B}\vec{\mathbf{c}}$$

Expressing $\vec{\mathbf{w}}$ in alternative coordinates: General method

Suppose $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is a basis of a vector space V and the basis vectors are written in **Cartesian coordinates.**

Consider a vector $\vec{\mathbf{w}}$ with **Cartesian coordinates $\vec{\mathbf{x}}$.**

If $\vec{\mathbf{w}}$ is in $V = span(B)$, then there must be unique coeffients $\vec{\mathbf{c}} = [c_1, c_2, \ldots, c_k]^T$ such that:

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{x}}$$

These coefficients will be the **alternative coordinates** of $\vec{\mathbf{w}}$ with respect to B.

Therefore, as we learned in Lecture 22 and Module 42, finding **alternative coordinates** of $\vec{\mathbf{w}}$ with respect to B boils down to solving a system of linear equations. Since B was assumed to be a basis, if this system is consistent, that is, if $\vec{\mathbf{w}}$ is in fact in V, then this system will have a unique solution.

Expressing $\vec{\mathbf{w}}$ in alternative coordinates when $k = n$

Suppose $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ is a basis of the *entire space* $\mathbb{R}^n$ for some n and the basis vectors are written in **Cartesian coordinates.**

Let $\vec{\mathbf{w}}$ be a vector in $\mathbb{R}^n$ with **Cartesian coordinates $\vec{\mathbf{x}}$.**

Then there are unique coefficients $\vec{\mathbf{c}} = [c_1, c_2, \dots, c_n]^T$ such that:

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n = \vec{\mathbf{x}}$$

These coefficients will be the **alternative coordinates** of $\vec{\mathbf{w}}$ with respect to B.

Finding these **alternative coordinates $\vec{\mathbf{c}}$** of $\vec{\mathbf{w}}$ with respect to B boils down to solving a system of linear equations. Since B was assumed to be a basis of $\mathbb{R}^n$, this system will always have a unique solution. If we write the basic vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ in the given order as the columns of a matrix $\mathbf{B}$, then $\mathbf{B}$ will be invertible, and this unique solution is given by

$$\vec{\mathbf{c}} = \mathbf{B}^{-1}\vec{\mathbf{x}}$$

Take-home message: Bases and parametrization

For a given n, we let $\vec{\mathbf{e}}_i$ denote the vector in $\mathbb{R}^n$ that has 1 in position i and 0 in all other positions. The set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n\}$ forms the *standard basis* of $\mathbb{R}^n$.

Its elements $\vec{\mathbf{e}}_i$ are called *standard basis vectors.*

Given any basis $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of a vector space V, for every vector $\vec{\mathbf{w}}$ in V there exists exactly one vector $\vec{\mathbf{c}} = [c_1, \ldots, c_k]^T$ of coefficients such that

$$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$$

These vectors $\vec{\mathbf{c}}$ give us *coordinates* for the elements of V and can be used to *parametrize* V.

When $V = \mathbb{R}^n$ and B is the standard basis, we get the **Cartesian coordinates**; for other bases B we get **alternative coordinates** with respect to B.

Take-home message: Changing bases

Suppose $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is a basis of a linear subspace V of some $\mathbb{R}^n$.

Let $\mathbf{B}$ be a matrix whose columns contain these basis vectors as columns in the given order, written in **Cartesian coordinates.**

Consider a vector $\vec{\mathbf{w}}$ in V with **Cartesian coordinates $\vec{\mathbf{x}}$** and **alternative coordinates $\vec{\mathbf{c}}$** $= [c_1, \ldots, c_k]^T$ with respect to B. Then

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{x}}$$

Thus we can compute $\vec{\mathbf{x}}$ from $\vec{\mathbf{c}}$ as the matrix product

$$\vec{\mathbf{x}} = \mathbf{B}\vec{\mathbf{c}}$$

We can find the $\vec{\mathbf{c}}$ by solving the above system of linear equations.

When $k = n$ and B is a basis for the entire space, then $\mathbf{B}$ is invertible and we can compute $\vec{\mathbf{c}}$ from $\vec{\mathbf{x}}$ as the matrix product

$$\vec{\mathbf{c}} = \mathbf{B}^{-1}\vec{\mathbf{x}}$$