Lecture 26: The Rank of a Matrix

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MATH3200: Applied Linear Algebra

Review of some terminology

Let V be a vector space. A set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ such that $V = span(S) = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$ is called a *spanning set* of V.

A linearly independent spanning set of V is called a *basis* of V.

If S is a spanning set of V, there is always a basis B of V that contains only vectors from S. Such a basis must be a *maximal linearly independent subset* of S.

Every two bases for the same vector space V have the same size. This size is called the *dimension* of V. It will be denoted $dim(V)$.

Review of notation : Columns and rows of a matrix

Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

Here we will use the following notation for the columns and rows of **A**:

The columns are $m \times 1$ column vectors

$$\vec{\mathbf{a}}_{*1} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad \vec{\mathbf{a}}_{*2} = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \qquad \dots \qquad \vec{\mathbf{a}}_{*n} = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The rows are $1 \times n$ row vectors

$$\vec{\mathbf{a}}_{1*} = [a_{11}, \dots, a_{1n}] \quad \vec{\mathbf{a}}_{2*} = [a_{21}, \dots, a_{2n}] \quad \dots \quad \vec{\mathbf{a}}_{m*} = [a_{m1}, \dots, a_{mn}]$$

The row space and the column space of a matrix

Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

Let $RS(\mathbf{A}) = span(\vec{\mathbf{a}}_{1*}, \vec{\mathbf{a}}_{2*}, \dots, \vec{\mathbf{a}}_{m*})$ be the linear span of all rows of $\mathbf{A}$. This space is called the *row space of* $\mathbf{A}$.

Similarly, let $CS(\mathbf{A}) = span(\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \dots, \vec{\mathbf{a}}_{*n})$ be the linear span of all columns of $\mathbf{A}$. This space is called the *column space of* $\mathbf{A}$.

Note that $RS(\mathbf{A})$ consists of $1 \times n$ row vectors and $CS(\mathbf{A})$ consists of $m \times 1$ column vectors.

In particular, $RS(\mathbf{A}) = CS(\mathbf{A})$ *only if* $\mathbf{A}$ is of order 1×1.

The row space and the column space: An example

Consider an 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Question L26.1: What are the row space $RS(\mathbf{A})$ and the column space $CS(\mathbf{A})$ of this matrix?

The row space $RS(\mathbf{A})$ consists of all linear combinations

$c_1[1,0,0] + c_2[0,0,1] + c_3[0,0,0] = [c_1, 0, c_2]$

It is the x-z-plane in the space of all 3-dimensional row vectors.

The column space $CS(\mathbf{A})$ consists of all linear combinations

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_3 \\ 0 \end{bmatrix}$$

It is the x-y-plane in the space of all 3-dimensional column vectors.

How about the dimensions of $RS(\mathbf{A})$ and $CS(\mathbf{A})$?

Let $\mathbf{A}$ be an $m \times n$ matrix.

Since the set of all rows $\{\vec{\mathbf{a}}_{1*}, \vec{\mathbf{a}}_{2*}, \ldots, \vec{\mathbf{a}}_{m*}\}$ of $\mathbf{A}$ is a spanning set for $RS(\mathbf{A})$, there must be a basis of $RS(\mathbf{A})$ that is contained in this set of row vectors. Such a basis must be maximal linearly independent subset of the set of all rows of $\mathbf{A}$.

Question L26.2: Can we have $dim(RS(\mathbf{A})) > m$?

No, because no subset of the set of rows of $\mathbf{A}$ can have more elements than the number m of rows of $\mathbf{A}$.

Similarly, since the set of all columns $\{\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \ldots, \vec{\mathbf{a}}_{*n}\}$ of $\mathbf{A}$ is a spanning set for $CS(\mathbf{A})$, there must be a basis of $CS(\mathbf{A})$ that is contained in this set of column vectors. Such a basis must be maximal linearly independent subset of the set of all columns of $\mathbf{A}$.

It follows that $dim(RS(\mathbf{A})) \leq m$ and $dim(CS(\mathbf{A})) \leq n$.

Row rank and column rank of a matrix $\mathbf{A}$

Definition

The *row rank* of a matrix $\mathbf{A}$ is the maximum size of a linearly independent subset of its row vectors.
The *column rank* of a matrix $\mathbf{A}$ is the maximum size of a linearly independent subset of its column vectors.

The row rank of a matrix $\mathbf{A}$ is equal to the dimension $dim(RS(\mathbf{A}))$ of its row space, and the column rank of a matrix $\mathbf{A}$ is equal to the dimension $dim(CS(\mathbf{A}))$ of its column space.

We have seen on the previous slide that for an $m \times n$ matrix $\mathbf{A}$ the row rank can be *at most* the number m of rows of $\mathbf{A}$ and the column rank can be *at most* the number n of columns of $\mathbf{A}$.

Row rank and column rank: First examples

Consider a zero matrix $\mathbf{O} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$.

Here each row and each column is a zero vector. Therefore the only linearly independent subsets of the sets of rows and columns are empty, of size zero.

It follows that both the row rank and the column rank of $\mathbf{O}$ are 0.

Question L26.3: Suppose the rows of an $m \times n$ matrix $\mathbf{A}$ form a linearly independent set. What is the row rank of $\mathbf{A}$?

The rows of an $m \times n$ matrix $\mathbf{A}$ form a linearly independent set, and only if, the row rank of $\mathbf{A}$ is m.

Similarly, the columns of an $m \times n$ matrix $\mathbf{A}$ form a linearly independent set if, and only if, the column rank of $\mathbf{A}$ is n.

An example of a 3×3 matrix in row echelon form

Consider a matrix of the form $\mathbf{A} = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Question L26.4: What is the row rank of $\mathbf{A}$?

The row rank is 1, since $\mathbf{A}$ contains exactly nonzero row.

Question L26.5: What is the column rank of $\mathbf{A}$?

The column rank is also 1, since the first column is a nonzero vector and each of the next two columns is a scalar multiple of the first:

$\vec{\mathbf{a}}_{*2} = a_{12}\vec{\mathbf{a}}_{*1}$ and $\vec{\mathbf{a}}_{*3} = a_{13}\vec{\mathbf{a}}_{*1}$.

Thus this matrix does not have two linearly independent columns.

Another example of a 3×3 matrix in echelon form

Consider a matrix of the form $\mathbf{A} = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$

Question L26.6: What is the row rank of **A**?

The row rank is 2, since **A** contains two nonzero rows, neither of which is a scalar multiple of the other. So the maximal number of linearly independent rows is 2.

Question L26.7: What is the column rank of **A**?

The column rank is also 2. The first two columns form a linearly independent set that is maximal, since the third column can be expressed as a linear combination of the first two columns:

$\vec{\mathbf{a}}_{*3} = (a_{13} - a_{12}a_{23})\vec{\mathbf{a}}_{*1} + a_{23}\vec{\mathbf{a}}_{*2}$.

The rank of 3×3 matrices in echelon form: The pattern

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Row rank = column rank = 1.

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$ Row rank = column rank = 2.

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ Row rank = column rank = 2.

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$ Row rank = column rank = 3.

For each of these row-reduced matrices, both the row rank and the column rank are equal to the number of *pivotal columns,* that is, columns that contain a first nonzero element of some row.

Gaussian elimination preserves row rank and column rank

Theorem

If $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by an elementary row operation, then the row rank of* $\mathbf{B}$ *is equal to the row rank of* $\mathbf{A}$ *and the column rank of* $\mathbf{B}$ *is equal to the column rank of* $\mathbf{A}$*.*

The proof of this theorem is a bit tedious; we will omit it here. The result is of interest to us mainly because of its following consequence:

Corollary

The process of Gaussian elimination preserves both row rank and column rank.

The rank of a matrix

The examples on slide 11 illustrate the following result:

Theorem

Both the row rank and the column rank of a matrix in generalized row echelon form are equal to its number of nonzero rows, which is equal to the number of pivotal columns.

Since Gaussian elimination does not change the row rank or column rank, we can deduce the following more general result:

Corollary

The row rank and the column rank of every matrix $\mathbf{A}$ *are equal.*

We call this common number the *rank* of $\mathbf{A}$. It is denoted by $r(\mathbf{A})$.

A procedure for finding $r(\mathbf{A})$

For any matrix $\mathbf{A}$, we can find $r(\mathbf{A})$ by:

1. performing enough steps of Gaussian elimination until we reach a matrix in generalized row-echelon form,
2. counting the number of nonzero rows or the number of pivotal columns in this resulting matrix. This number is the rank of the original matrix $\mathbf{A}$.

Recall that the *generalized row echelon form* is like the row-echelon form, but we don't require that the leading elements of the nonzero rows must be 0.

Also recall that a *pivotal column* is a column that contains the leading (that is, first non-zero) element of some nonzero row.

An example of the procedure for finding $r(\mathbf{A})$

Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 14 & -83 & 22 & -7 \\ 0 & 0 & 0 & 31 & -25 \\ 0 & 28 & -166 & 13 & 11 \end{bmatrix}$

Perform Gaussian elimination:

$$\xrightarrow{R3 \mapsto R3 - 2R1} \begin{bmatrix} 0 & 14 & -83 & 22 & -7 \\ 0 & 0 & 0 & 31 & -25 \\ 0 & 0 & 0 & -31 & 25 \end{bmatrix}$$

$$\xrightarrow{R3 \mapsto R3 + R2} \begin{bmatrix} 0 & 14 & -83 & 22 & -7 \\ 0 & 0 & 0 & 31 & -25 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have obtained a matrix in generalized row-echelon form. Its pivotal columns are column number 2 and column number 4. Thus we can conclude that the rank of this matrix and of our original one is $r(\mathbf{A}) = 2$.

Matrices of full rank

Square matrices of maximal possible rank are said to be of *full rank.*

Definition

An $n \times n$ square matrix $\mathbf{A}$ is said to have *full rank* if $r(\mathbf{A}) = n$, that is, if its column vectors (equivalently: its row vectors) form a linearly independent set.

Only square matrices can have full rank. But the notion can be used to characterize the rank of any matrix.
We give this characterization here FYI only:

Theorem

The rank $r(\mathbf{A})$ of a matrix is the largest n such that $\mathbf{A}$ contains a submatrix $\mathbf{B}$ of order $n \times n$ and full rank.

Take-home message

The *row space* $RS(\mathbf{A}) = span(\vec{\mathbf{a}}_{1*}, \vec{\mathbf{a}}_{2*}, \ldots, \vec{\mathbf{a}}_{m*})$ of a matrix $\mathbf{A}$ is the linear span of all of its rows.

The *column space* $CS(\mathbf{A}) = span(\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \ldots, \vec{\mathbf{a}}_{*n})$ of a matrix $\mathbf{A}$ is the linear span of all of its columns.

$RS(\mathbf{A})$ and $CS(\mathbf{A})$ are different vector spaces, but must have the same dimension $dim(RS(\mathbf{A})) = dim(CS(\mathbf{A})) = r(\mathbf{A})$, called the *rank of* $\mathbf{A}$.

The rank $r(\mathbf{A})$ is equal to the maximum size of a linearly independent subset of its rows, aka the *row rank of* $\mathbf{A}$, and is also equal to the the maximum size of a linearly independent subset of its columns, aka as the *column rank of* $\mathbf{A}$.

Take-home message, continued

The rank of a matrix in generalized row echelon form is equal to the number of its nonzero rows, and is also equal to the number of its *pivotal columns,* that is, columns that contain a first nonzero element of some row.

Gaussian elimination preserves the rank of a matrix. It can be used to determine the rank of a matrix by finding an equivalent matrix in generalized row echelon form and counting its pivotal columns.

An $n \times n$ square matrix $\mathbf{A}$ is said to have *full rank* if $r(\mathbf{A}) = n$, that is, if its column vectors (equivalently: its row vectors) form a linearly independent set.

It is sometimes useful to know that the rank $r(\mathbf{A})$ of any matrix $\mathbf{A}$ is the largest n such that $\mathbf{A}$ contains a submatrix $\mathbf{B}$ of order $n \times n$ and full rank.