Lecture 27: The Rank and Consistency of Systems of Linear Equations

Winfried Just
Department of Mathematics, Ohio University

MATH3200: Applied Linear Algebra

Review of notation and terminology

Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

The columns are $m \times 1$ column vectors

$$\vec{\mathbf{a}}_{*1} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad \vec{\mathbf{a}}_{*2} = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \qquad \dots \qquad \vec{\mathbf{a}}_{*n} = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$CS(\mathbf{A}) = span(\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \dots, \vec{\mathbf{a}}_{*n})$ is the column space of $\mathbf{A}$.

Review: Consistency of systems of linear equations

Let $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ be the matrix form of a system of linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The following statements are equivalent, that is, express the same property in different ways:

- The system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent.
- $\vec{\mathbf{b}} = x_1\vec{\mathbf{a}}_{1*} + x_2\vec{\mathbf{a}}_{2*} + \cdots + x_n\vec{\mathbf{a}}_{n*}$ is a linear combination of the column vectors $\vec{\mathbf{a}}_{1*}, \vec{\mathbf{a}}_{2*}, \ldots, \vec{\mathbf{a}}_{n*}$ of $\mathbf{A}$. Here the vector of coefficients $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]^T$ must be a solution of the above linear system.
- $\vec{\mathbf{b}}$ is in $span(\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \ldots, \vec{\mathbf{a}}_{*n}) = CS(\mathbf{A})$.

Consistency and the rank

Let $\mathbf{A}\vec{x} = \vec{\mathbf{b}}$ be a system of linear equations. Consider the coefficient matrix $\mathbf{A}$ and the augmented matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ of this system:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}$$

By the information on the previous slide, the system $\mathbf{A}\vec{x} = \vec{\mathbf{b}}$ is consistent if, and only if, the last column of $[\mathbf{A}, \vec{\mathbf{b}}]$ is in the column space $CS(\mathbf{A})$ of $\mathbf{A}$.

Question L27.1: If the system $\mathbf{A}\vec{x} = \vec{\mathbf{b}}$ is consistent, what can we say about the relation between $r(\mathbf{A})$ and $r([\mathbf{A}, \vec{\mathbf{b}}])$?

Consistency and the rank: A theorem

When the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent, then the last column of $[\mathbf{A}, \vec{\mathbf{b}}]$ must be a linear combination of the columns of the coefficient matrix $\mathbf{A}$. Thus every maximal linearly independent subset of the columns of $\mathbf{A}$ must remain a maximal linearly independent subset of $CS([\mathbf{A}, \vec{\mathbf{b}}])$. More generally:

Theorem

The system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent if, and only if, $r(\mathbf{A}) = r([\mathbf{A}, \vec{\mathbf{b}}])$.

Let us illustrate this theorem on the next slides with two simple examples where the extended matrices are in row echelon form.

Example 1: A simple system

Consider the system of linear equations

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & 5x_3 & = & 6 \\ & & & & 0 & = & 1 \end{array}$$

Here the coefficient matrix and the extended matrix are:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question L27.2: Find $r(\mathbf{A})$ and $r([\mathbf{A}, \vec{\mathbf{b}}])$.

Here $r(\mathbf{A}) = 2 < 3 = r([\mathbf{A}, \vec{\mathbf{b}}])$, and the last column of $[\mathbf{A}, \vec{\mathbf{b}}]$ is not in $CS(\mathbf{A})$. The theorem tells us correctly that the system is inconsistent.

Example 2: Another simple system

Consider the system of linear equations

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & & & x_3 & = & 6 \end{array}$$

Here the coefficient matrix and the extended matrix are:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Question L27.3: Find $r(\mathbf{A})$ and $r([\mathbf{A}, \vec{\mathbf{b}}])$.

Here $r(\mathbf{A}) = 2 = r([\mathbf{A}, \vec{\mathbf{b}}])$. The theorem tells us correctly that the system is consistent.

Consistency and the rank: Another observation

Question L27.4: Let $\mathbf{A}$ be the coefficient matrix of the system for Example 2 on the previous slide. Is it true that for every 2×1 column vector $\vec{\mathbf{b}}$ the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ will be consistent?

Yes, because $r([\mathbf{A}, \vec{\mathbf{b}}])$ can never be larger than $r(\mathbf{A}) = 2$ when there are only 2 rows.

We can imediately deduce that, for example, each of the following systems with the same coefficient matrix must be consistent:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 0.4 \\ & & & & x_3 & = & 687.9 \end{array}$$

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & \ln 55 \\ & & & & x_3 & = & \pi^2 \end{array}$$

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & \sin 3 \\ & & & & x_3 & = & \arctan 77 \end{array}$$

Corollary

When $r(\mathbf{A}) = m$ is equal to the number of rows of $\mathbf{A}$, then every system of the form $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent.

Proof of the Corollary: Since $[\mathbf{A}, \vec{\mathbf{b}}]$ has also m rows, $r([\mathbf{A}, \vec{\mathbf{b}}]) \leq m$. And since every column of $\mathbf{A}$ is also a column of $[\mathbf{A}, \vec{\mathbf{b}}]$, we also must have $r(\mathbf{A}) \leq r([\mathbf{A}, \vec{\mathbf{b}}])$.
Thus $r(\mathbf{A}) = m$ implies that $r([\mathbf{A}, \vec{\mathbf{b}}]) = m = r(\mathbf{A})$. $\square$

When the rank of the coefficient matrix $\mathbf{A}$ of a consistent system is smaller than the number of its rows, then we cannot draw this conclusion. For example, the following consistent system has the same coefficient matrix as the inconsistent system in Example 1:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & 5x_3 & = & 6 \\ & & & & 0 & = & 0 \end{array}$$

Take-home message

In this lecture we saw two results that connect the rank of the coefficient and extended matrices of systems of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ with consistency of these systems.

Theorem

The system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is consistent if, and only if,* $r(\mathbf{A}) = r([\mathbf{A}, \vec{\mathbf{b}}])$.

Corollary

When $r(\mathbf{A}) = m$ *is equal to the number of rows of* $\mathbf{A}$*, then every system of the form* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is consistent.*