Leture 28: The Null Space of a Matrix

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MATH3200: Applied Linear Algebra

The null space of a matrix

Definition

Let $\mathbf{A}$ be an $m \times n$ matrix. The *null space* $N(\mathbf{A})$ of $\mathbf{A}$ is the set of all vectors $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ such that $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

In the literature, both spellings "null space" and "nullspace" are used, but the first is more common.

The dimension $dim(N(\mathbf{A}))$ is also sometimes called the *nullity of* $\mathbf{A}$, but we will not use this terminology here.

As the name null **space** suggests, $N(\mathbf{A})$ is always a linear subspace of $\mathbb{R}^n$. We will prove this in Module 52.

In order to find the null space of a given matrix, we solve the homogenous system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ by Gaussian elimination. The solution set is the null space. If it has more than one elements, we can describe it in terms of one or more variables that are being left as *free parameters.*

Example 1 of a null space

Let $\mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ This is an $m \times n$ matrix with $n = 2$.

Perform Gaussian elimination on the extended matrix $[\mathbf{A}_1, \vec{\mathbf{0}}]$ of the homogeneous system $\mathbf{A}_1\vec{\mathbf{x}} = \vec{\mathbf{0}}$:

$$[\mathbf{A}_1, \vec{\mathbf{0}}] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 2R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Note: By ignoring the last column we see that rank $r(\mathbf{A}_1) = 2$.

Thus $N(\mathbf{A}_1)$ consists of all solutions of the equivalent system

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 0 \\ & & x_2 & = & 0 \end{array}$$

Here we must have $x_1 = x_2 = 0$, and the solution set $N(\mathbf{A}_1)$ contains only the vector $\vec{\mathbf{0}} = [0, 0]^T$.

Note that $r(\mathbf{A}_1) + dim(N(\mathbf{A}_1)) = 2 + 0 = n$.

Example 2 of a null space

Let $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ This is an $m \times n$ matrix with $n = 3$.

Perform Gaussian elimination on the extended matrix $[\mathbf{A}_2, \vec{\mathbf{0}}]$ of the homogeneous system $\mathbf{A}_2\vec{\mathbf{x}} = \vec{\mathbf{0}}$:

$$[\mathbf{A}_2, \vec{\mathbf{0}}] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 2R1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Question L28.1: What is $r(\mathbf{A}_2)$?

When we remove the last column, we obtain the row echelon form of the matrix $\mathbf{A}_2$. It has 2 nonzero rows and 2 pivotal columns (columns 1 and 3), and it follows that the rank $r(\mathbf{A}_2) = 2$.

Here $N(\mathbf{A}_2)$ consists of all solutions of the equivalent system

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

Example 2 of a null space, continued

Let $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ This is an $m \times n$ matrix with $n = 3$.

Here $N(\mathbf{A}_2)$ consists of all solutions of the equivalent system

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

Question L28.2: What is the solution set $N(\mathbf{A}_2)$?

We must have $x_3 = 0$, and then we can choose any of the other variables x_1, x_2 as our free parameter to represent the vectors in the solution set $N(\mathbf{A}_2)$.

For example, if we choose x_1 as our free parameter and substitute $x_3 = 0$ in the first equation, then we obtain the solution set as the set of all vectors of the form $[x_1, -x_1, 0]^T$.

Alternatively, we could choose x_2 as our free parameter and obtain the solution set as the set of all vectors of the form $[-x_2, x_2, 0]^T$.

Example 2 of a null space, completed

Let $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ This is an $m \times n$ matrix with $n = 3$.

We can represent $N(\mathbf{A}_2)$ as the set of all vectors $\begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix}$

Now let us find a basis for $N(\mathbf{A}_2)$. We can do this by setting the free variable to 1.

We get exactly one basis vector, so that

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$ is a basis.

Note that $r(\mathbf{A}_2) + dim(N(\mathbf{A}_2)) = 2 + 1 = 3 = n$.

Example 3 of a null space

Let $\mathbf{A}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix}$ This is an $m \times n$ matrix with $n = 4$.

Perform Gaussian elimination on the extended matrix $[\mathbf{A}_3, \vec{\mathbf{0}}]$ of the homogeneous system $\mathbf{A}_3\vec{\mathbf{x}} = \vec{\mathbf{0}}$:

$$[\mathbf{A}_3, \vec{\mathbf{0}}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 2R1} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Note: $r(\mathbf{A}_3) = r([\mathbf{A}_3, \vec{\mathbf{0}}]) = 2.$

Here $N(\mathbf{A}_3)$ consists of all solutions of the equivalent system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \\ & & & & x_3 & & & = & 0 \end{array}$$

Example 3 of a null space, continued

Let $\mathbf{A}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix}$ This is an $m \times n$ matrix with $n = 4$.

Here $N(\mathbf{A}_3)$ consists of all solutions of the equivalent system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \\ & & & & x_3 & & & = & 0 \end{array}$$

Question L28.3: What is the solution set $N(\mathbf{A}_3)$?

Again we must have $x_3 = 0$, and then we can choose any two of the other variables x_1, x_2, x_4 as our free parameters to represent the vectors in the solution set $N(\mathbf{A}_3)$. For example, if we choose x_1 and x_4 as our free parameters and substitute $x_3 = 0$ in the first equation, then we obtain the solution set as the set of all vectors of the form $[x_1, -x_1 - x_4, 0, x_4]^T$.

Example 3 of a null space, completed

Let $\mathbf{A}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix}$ This is an $m \times n$ matrix with $n = 4$.

We can represent $N(\mathbf{A}_2)$ as the set of all vectors $\begin{bmatrix} x_1 \\ -x_1 - x_4 \\ 0 \\ x_4 \end{bmatrix}$

Now let us find a basis for $N(\mathbf{A}_3)$. We can do this by successively setting each free variable to 1 and all other free variables to 0.

For our choice of free variables, this gives the following basis:

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis.}$$

Note that $r(\mathbf{A}_3) + dim(N(\mathbf{A}_3)) = 2 + 2 = 4 = n$.

The dimension of $N(\mathbf{A})$ and $r(\mathbf{A})$

The examples on the preceding slides suggest that there is a tight relationship between the rank $r(\mathbf{A})$ of a matrix, the dimension $dim(N(\mathbf{A}))$ of its null space, and the number n of columns of $\mathbf{A}$.

The following theorem shows that in fact there is such a relationship:

Theorem

Let $\mathbf{A}$ *be an* $m \times n$ *matrix. Then*

$$r(\mathbf{A}) + dim(N(\mathbf{A})) = n.$$

Question L28.4: Suppose $\mathbf{A}$ is a 4×5 matrix such that $N(\mathbf{A})$ has a basis B that contains 2 vectors. What is $r(\mathbf{A})$?

Here $n = 5$ and the information on the basis of the null space of $\mathbf{A}$ shows that $dim(N(\mathbf{A}) = 2$.

Thus $r(\mathbf{A}) = n - dim(N(\mathbf{A})) = 5 - 2 = 3$.

Take-home message

Consider a linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ with coefficient matrix $\mathbf{A}$ of order $m \times n$.

- The *null space* of $\mathbf{A}$ is the set of vectors $\vec{\mathbf{x}}$ such that $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
- The null space consists of column vectors and is a linear subspace of $\mathbb{R}^n$.
- The dimension of the null space is $dim(N(\mathbf{A})) = n - r(\mathbf{A})$.
- In order to find the null space and a basis for it, we can proceed as follows:
  1. Solve the homogenous system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
  2. If there is a unique solution, then $N(\mathbf{A}) = \{\vec{\mathbf{0}}\}$ and the empty set is the basis.
  3. Otherwise the solution set can be described in terms of choosing $k = dim(N(\mathbf{A}))$ among the variables as *free parameters.* We obtain a set of basis vectors for $N(\mathbf{A})$ by successively setting each of these free parameters to 1 while setting all other free parameters to 0.