Lecture 29: The Rank and Theory of Solutions
How to Represent Solution Sets

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MATH3200: Applied Linear Algebra

Two theorems

Let us explicitly state as theorems two results that were discussed in Conversation 29:

Theorem

Consider a linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *with* $\mathbf{A}$ *of order* $m \times n$*.*

- *When* $r(\mathbf{A}) = m$*, the system is always consistent.*
- *When* $r(\mathbf{A}) < m$*, the system is consistent for some, but not for all choices of* $\vec{\mathbf{b}}$*.*

Theorem

Consider a linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *with* $\mathbf{A}$ *of order* $m \times n$*.*

- *When* $r(\mathbf{A}) = n$*, the system is either inconsistent or has exactly one solution.*
- *When* $r(\mathbf{A}) < n$*, the system is either inconsistent or has infinitely many solutions.*

One more theorem

Theorem

Suppose $\mathbf{A}$ *is the coefficient matrix of a linear system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Let $\vec{\mathbf{x}}$ *be a solution of this system and let* $\vec{\mathbf{y}}$ *be another vector. Then* $\vec{\mathbf{y}}$ *is also a solution of the same system if, and only if,* $\vec{\mathbf{x}} - \vec{\mathbf{y}}$ *is in* $N(\mathbf{A})$.

In other words, when $\vec{\mathbf{x}}$ *is one solution of the system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, *then all other solutions must be of the form* $\vec{\mathbf{x}} + \vec{\mathbf{z}}$, *where* $\vec{\mathbf{z}}$ *is in* $N(\mathbf{A})$.

It follows that when $\mathbf{A}$ is of order $m \times n$, the solution set of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ can always be written as the set of all vectors of the form $\vec{\mathbf{x}} + \vec{\mathbf{z}}$, where $\vec{\mathbf{z}}$ belongs to a vector space of dimension $dim(N(\mathbf{A})) = n - r(\mathbf{A})$.

Question L29.1: Suppose $\mathbf{A}$ is a 3×5 matrix with $r(\mathbf{A}) = 3$. Then $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is always consistent. How does the solution set look like?

Here $dim(N(\mathbf{A})) = 5 - 3 = 2$, so that the solution set will be a plane. It will be a plane through the origin if, and only if, $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ so that $\vec{\mathbf{0}}$ is a solution. Note that $\vec{\mathbf{b}} \neq 0$, then $\mathbf{A}\vec{\mathbf{0}} = \vec{\mathbf{0}} \neq \vec{\mathbf{b}}$, so that $\vec{\mathbf{0}}$ is not a solution of the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and will not lie in this plane.

Example 1

Consider a system of the form:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & b_1 \\ & & & & x_3 & = & b_2 \end{array}$$

Here the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Since $r(\mathbf{A}) = 2 = m$, the system is consistent for every choice of $\vec{\mathbf{b}}$.

Here the extended matrix of the system is already in row-reduced form and we can solve the system by back-substitution, choosing one of the variables x_1, x_2 as our *free parameter.*

Question L29.2: What do we get when we choose x_1 as our free parameter?

The solution set consists of all vectors $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ b_1 - b_2 - x_1 \\ b_2 \end{bmatrix}$

Example 1, completed

Consider a system of the form:
$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & b_1 \\ & & & & x_3 & = & b_2 \end{array}$$

Here the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$dim(N(\mathbf{A})) = 1$ and $N(\mathbf{A})$ consists of all vectors $\begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix}$.

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis of $N(\mathbf{A})$. The solution set consists of all

vectors $\begin{bmatrix} 0 \\ b_1 - b_2 \\ b_2 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ where $\begin{bmatrix} 0 \\ b_1 - b_2 \\ b_2 \end{bmatrix}$ is one solution.

Example 2

Consider a system of the form:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & b_1 \\ & & & & 0 & = & b_2 \end{array}$$

Here the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Since $r(\mathbf{A}) = 1 < m$, the system is consistent only for some $\vec{\mathbf{b}}$. More precisely, the system is consistent only when $b_2 = 0$.

If $b_2 = 0$, we can solve the system by back-substitution, choosing 2 among the variables x_1, x_2, x_3 as our *free parameters.*

When we choose, for example, x_1 and x_3 as free parameters we find that

the solution set consists of all vectors $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ b_1 - x_1 - x_3 \\ x_3 \end{bmatrix}$

Example 2, completed

Consider a system of the form:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & b_1 \\ & & & & 0 & = & 0 \end{array}$$

Here the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Question L29.3: What are $dim(N(\mathbf{A}))$ and $N(\mathbf{A})$?

$dim(N(\mathbf{A})) = 2$ and $N(\mathbf{A})$ consists of all vectors $\begin{bmatrix} x_1 \\ -x_1 - x_3 \\ x_3 \end{bmatrix}$.

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis of $N(\mathbf{A})$. The solution set

consists of all vectors $\begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Take-home message: What do these examples illustrate?

In both examples we had $r(\mathbf{A}) < n$. In Example 1 we needed $dim(N(\mathbf{A})) = 1$ free parameter and in Example 2 we needed $dim(N(\mathbf{A})) = 2$ free parameters to describe the solution set.

Theorem

Suppose $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is a consistent linear system with a coefficient matrix of order $m \times n$. Then the solution set can be described by choosing exactly $k = dim(N(\mathbf{A})) = n - r(\mathbf{A})$ among the variables $x_1, \dots, x_n$ as free parameters.

In both examples we could pick one particular solution $\vec{\mathbf{x}}$ and a basis $B = \{\vec{\mathbf{z}}_1, \dots, \vec{\mathbf{z}}_k\}$ of $N(\mathbf{A})$ so that all solution vectors could be written in the form $\vec{\mathbf{x}} + c_1\vec{\mathbf{z}}_1 + \cdots + c_k\vec{\mathbf{z}}_k$ for some coefficients. This will always be possible in view of the theorem on slide 3.

When the basis B of $N(\mathbf{A})$ is constructed in the way that we learned in Lecture 28, we can choose our free variables x_i as coefficients c_j here. But note that c_2 for Example 2 would be x_3.