Lecture 2: More Examples of Matrices: Vectors

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MATH3200: Applied Linear Algebra

Vectors

A $1 \times n$ matrix $[a_{11}, a_{12}, \ldots, a_{1n}]$ is also called a *row vector.*

An $m \times 1$ matrix $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ is also called a *column vector.*

For example, the coordinates x, y, z of a point in three-dimensional Euclidean space allow us to specify the position of a point as a 1×3 row vector $[x, y, z]$ or the position of a moving object at time t as a 1×4 row vector $[x, y, z, t]$.

The same information can also be represented as 3×1 or 4×1 column vectors, respectively.

Question L2.1: Which representation is the correct one?

There is *nothing* that makes position *inherently* a row vector or a column vector. The choice of representation is dictated by the *particular calculations* that we want to perform on these vectors.

Notation and terminology

Vectors are special kinds of matrices, but we will use a different notation for them:

$\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}}, \vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$.

This is also the notation that you should use in your notes, quizzes, and tests.

When $\vec{\mathbf{x}}$ is a $1 \times n$ row vector or an $n \times 1$ column vector, then n is usually called the *dimension* or simply the *length* of the vector $\vec{\mathbf{x}}$.

Columns of a matrix

Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

The columns are $m \times 1$ column vectors

$$\vec{\mathbf{a}}_{*1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \vec{\mathbf{a}}_{*2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \quad \vec{\mathbf{a}}_{*n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

A matrix of column vectors

Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n} \qquad \text{and let}$$

$\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \dots, \vec{\mathbf{a}}_{*n}$ be defined as on the previous slide.

Then $\mathbf{A}_{cols} = [\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \dots, \vec{\mathbf{a}}_{*n}]$

is a $1 \times n$ matrix *whose elements are column vectors.*

For example, if $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $\mathbf{A}_{cols} = \begin{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{bmatrix}$

Are $\mathbf{A}$ and $\mathbf{A}_{cols}$ the same matrix?

Definition

Two matrices $[a_{ij}]_{m \times n}$ and $[b_{ij}]_{m' \times n'}$ *are equal* if they have the same order (that is, $m = m'$ and $n = n'$), and $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Question L2.2: So are $\mathbf{A}$ and $\mathbf{A}_{cols}$ the same matrix?

The matrices $\mathbf{A}$ and $\mathbf{A}_{cols}$ are *not equal.*

They have different orders, and the entries of $\mathbf{A}$ may be numbers, while the entries of $\mathbf{A}_{cols}$ are always vectors.

Thus in view of the above definition, $\mathbf{A} \neq \mathbf{A}_{cols}$.

But $\mathbf{A}$ and $\mathbf{A}_{cols}$ give us *the same information,* albeit organized in a different way. We will consider them *equivalent.*

Summary: Matrices and vectors

- Two matrices $[a_{ij}]_{m \times n}$ and $[b_{ij}]_{m' \times n'}$ *are equal* if they have the same order (that is, $m = m'$ and $n = n'$), and $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.
- A $1 \times n$ matrix is called a *row vector* and $m \times 1$ is also called a *column vector*.
- Vectors will be denoted by symbols like: $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}}, \vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$.
- The number of entries in a vector is called its *dimension* or its *length*.
- Linear arrays of data can be represented as either row vectors or column vectors; it depends on the particular calculation that we want to perform which representation to use.