Lecture 31: Introduction to Determinants
Definition, Examples, and Basic Properties

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MATH3200: Applied Linear Algebra

Determinants: Notation

Let $\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ be a square matrix.

The *determinant* of $\mathbf{A}$ is a number associated with $\mathbf{A}$.

It is denoted by $\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$ or $\det(\mathbf{A})$.

- Note that it matters a lot whether we enclose the elements of a matrix in square brackets, which signifies a matrix, or in straight lines, which signifies a single number.
- Only square matrices have determinants.
- Alternatively, one can think of the determinant as a function det that assigns a number $\det(\mathbf{A})$ to every square matrix $\mathbf{A}$.

Determinants of 1×1 matrices

When $\mathbf{A} = [a_{11}]$ has order 1×1, then we define:

$\det(\mathbf{A}) = \det([a_{11}]) = a_{11}$.

In this case we do *not* use the notation $|a_{11}|$ for the determinant, because it conflicts with the notation for the absolute value.

For example, $|-43| = 43$, while $\det([-43]) = -43$.

Determinants of 2×2 matrices

When $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has order 2×2, then we define:

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

It may be easier to memorize and use this formula as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Question L31.1: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ Find $\det(\mathbf{A})$.

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = (1)(3) - (2)(4) = -5.$$

Determinants of 3×3 matrices

When $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ has order 3×3, then we define:

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Question L32.2: Do you want to memorize the last formula?

Do you want to see and memorize the formula for order 4×4?

Defining determinants: Alternative approaches

It is *possible* to give explicit formulas for $\det(\mathbf{A})$ along the lines of the ones on the previous slides for all orders $n \times n$.

But this approach is *impractical.* These formulas are difficult to understand for $n > 3$ and difficult to use.

Textbooks often take the different approach of introducing a procedure for calculating determinants, and then defining $\det(\mathbf{A})$ as the number that one gets after performing these calculations. This approach is *more practical,* but the procedure itself is rather *not intuitive.*

Here we will take a third approach: Define det as a function that assigns numbers $\det(\mathbf{A})$ to square matrices and satisfies *fairly intuitive properties.* Then we will show a *fairly easy and practical* way of calculating $\det(\mathbf{A})$ based on these properties.

Property 1: Switching rows

Example: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

Then $\det(\mathbf{A}) = -5$ and $\det(\mathbf{B}) = (4)(2) - (3)(1) = 5 = -\det(\mathbf{A})$.

Proposition

When you switch the order of two rows of a square matrix, the determinant retains its absolute value but switches its sign.

Proof for 2×2 matrices:

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Switch rows: $\mathbf{B} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

Then $\det(\mathbf{A}) = ad - bc$, and $\det(\mathbf{B}) = cb - da = -\det(\mathbf{A})$. $\square$

Property 2: Multiplying one row by a scalar

Example: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 4 & 3 \end{bmatrix}$

Then $\det(\mathbf{A}) = -5$; $\det(\mathbf{B}) = (3)(3) - (6)(4) = -15 = 3\det(\mathbf{A})$.

Proposition

When you multiply one row of a square matrix by a scalar λ, the determinant changes by a factor of λ.

Proof for 2×2 matrices: Let λ be a scalar,

and let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Multiply row 1 by λ: $\mathbf{B} = \begin{bmatrix} \lambda a & \lambda b \\ c & d \end{bmatrix}$

Then $\det(\mathbf{B}) = \lambda ad - \lambda bc = \lambda(ad - bc) = \lambda \det(\mathbf{A})$.

Question L31.3: What is still missing in this proof?

We need to add a sentence like: "The proof for the case when we multiply the second row by λ is analogous." $\square$

Property 3: Adding a scalar multiple of one row to another

Example: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix}$ Here **B**

is obtained by adding 2 times the first row of **A** to its second row.

Then $\det(\mathbf{A}) = -5$ and $\det(\mathbf{B}) = (1)(7) - (2)(6) = -5 = \det(\mathbf{A})$.

Proposition

When you add a scalar multiple of one row of a square matrix to another row, the determinant does not change.

Proof for 2×2 matrices: Let λ be a scalar, and let

$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Add λ(row 1) to row 2: $\mathbf{B} = \begin{bmatrix} a & b \\ c + \lambda a & d + \lambda b \end{bmatrix}$

Then $\det(\mathbf{B}) = a(d + \lambda b) - b(c + \lambda a)$

$= (ad - bc) + (a\lambda b - b\lambda a) = \det(\mathbf{A}) + 0 = \det(\mathbf{A})$.

The proof for the case when we add a scalar multiple of the second row to the first row is analogous. $\square$

Property 4: Determinants of triangular matrices

Example: Let $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

Here **A** is upper-triangular and **B** is lower triangular.
$\det(\mathbf{A}) = (2)(4) - (3)(0) = 8$ and $\det(\mathbf{B}) = (2)(3) - (0)(4) = 6$.

Proposition

The determinant of a (lower- or upper) triangular square matrix is the product of the elements on its (main) diagonal.

Proof for 2×2 matrices: Each 2×2 triangular matrix is of the form:

$$\mathbf{A} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad \text{or} \quad \mathbf{B} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

Then $\det(\mathbf{A}) = ad - b(0) = ad$ and $\det(\mathbf{B}) = ad - (0)c = ad$. $\square$

Our definition of the determinant

Now we turn these properties into a definition of a function:

Definition

The function det assigns to *every square matrix* a number $\det(\mathbf{A})$ and has the following properties:

1. When $\mathbf{B}$ is obtained by switching two rows of $\mathbf{A}$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.
2. When $\mathbf{B}$ is obtained by multiplying of one row of $\mathbf{A}$ by a scalar λ, then $\det(\mathbf{B}) = \lambda \det(\mathbf{A})$.
3. When $\mathbf{B}$ is obtained by adding a scalar multiple of one row of $\mathbf{A}$ to another row, then $\det(\mathbf{B}) = \det(\mathbf{A})$.
4. If $\mathbf{A}$ is upper-triangular or lower-triangular, then $\det(\mathbf{A})$ is the product of the diagonal elements.

"The" suggests that there is exactly one such function. *Is there?*

Take-home message

To every square matrix **A** we can assign a number det(**A**) called its *determinant.*

We will see soon that det(**A**) gives us important information about **A** and that determinants have many important applications.

While it is possible to define determinants by explicit formulas, You will be required to memorize only the formulas for 1×1 and 2×2 matrices. For higher orders, our definition specifies det(**A**) in terms on four important properties that we have proved for 2×2 matrices. This is a legitimate definition, because there is only one function that has all of these properties.

Question L31.4: How could we convince ourselves that there is only one such function?

In the next lecture, you will see an explicit procedure for calculating determinants based on our definition. Since the definition allows us to unambiguously calculate det(**A**) for all square matrices **A**, there can only be one function with these properties.