Lecture 32: Calculating Determinants by Pivotal Condensation

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MATH3200: Applied Linear Algebra

Review: Our definition of the determinant

Definition

The function det assigns to *every square matrix* a number det($\mathbf{A}$) and has the following properties:

1. When $\mathbf{B}$ is obtained by switching two rows of $\mathbf{A}$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.
2. When $\mathbf{B}$ is obtained by multiplying of one row of $\mathbf{A}$ by a scalar λ, then $\det(\mathbf{B}) = \lambda \det(\mathbf{A})$.
3. When $\mathbf{B}$ is obtained by adding a scalar multiple of one row of $\mathbf{A}$ to another row, then $\det(\mathbf{B}) = \det(\mathbf{A})$.
4. If $\mathbf{A}$ is upper-triangular or lower-triangular, then $\det(\mathbf{A})$ is the product of the diagonal elements.

In this lecture we will illustrate a procedure, called *pivotal condensation,* that allows us to compute $\det(\mathbf{A})$ for every square matrix $\mathbf{A}$ based on these properties.

Our definition of the determinant in terms of elementary row operations

When we use the notation **EA** for the matrix obtained by applying an elementary row operation to **A**, our definition becomes:

Definition

The function det assigns to *every square matrix* a number $\det(\mathbf{A})$ and has the following properties:

1. If **E** implements elementary row operation (E1): "Exchange two rows of **A**," then $\det(\mathbf{EA}) = -\det(\mathbf{A})$.
2. If **E** implements elementary row operation (E2): "Multiply one row of **A** by a scalar $\lambda \neq 0$," then $\det(\mathbf{EA}) = \lambda \det(\mathbf{A})$.
3. If **E** implements elementary row operation (E3): "Add a scalar multiple of one row of **A** to another row of **A**," then $\det(\mathbf{EA}) = \det(\mathbf{A})$.
4. If **A** is upper-triangular or lower-triangular, then $\det(\mathbf{A})$ is the product of the diagonal elements.

(How) can we calculate det(**A**) from these properties?

1. Transform **A** by successive elementary row operations into
   $$\mathbf{A} \to \mathbf{E}_1\mathbf{A} \to \mathbf{E}_2\mathbf{E}_1\mathbf{A} \to \cdots \to \mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U},$$
   where **U** is upper-triangular.
2. Use Properties 1–3 of the definition on the previous slide to keep track of how the determinant changes at every step.
3. Calculate det(**U**) as the product of the diagonal elements.
4. Deduce det(**A**) from the results you obtained in the previous steps.

The procedure outlined above is called *pivotal condensation.*

It always gives you a unique number for det(**A**), and we can see that the properties on the previous slides uniquely determine the values of the function det.

Pivotal condensation: An example

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & -0.4 & 1 \\ -1 & 4 & 9 \end{bmatrix}$ Form $\mathbf{E}_1\mathbf{A}$ by switching rows 1 and 2:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & -0.4 & 1 \\ -1 & 4 & 9 \end{bmatrix} \longrightarrow \mathbf{E}_1\mathbf{A} = \begin{bmatrix} 0.2 & -0.4 & 1 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{bmatrix}$$

Question L32.1: How is $\det(\mathbf{A})$ related to $\det(\mathbf{E}_1\mathbf{A})$?

$\det(\mathbf{A}) = -\det(\mathbf{E}_1\mathbf{A})$.

An example of pivotal condensation, continued

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & -0.4 & 1 \\ -1 & 4 & 9 \end{bmatrix}$ $\quad \mathbf{E}_1\mathbf{A} = \begin{bmatrix} 0.2 & -0.4 & 1 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{bmatrix}$

Then $\det(\mathbf{A}) = -\det(\mathbf{E}_1\mathbf{A})$.

Form $\mathbf{E}_2\mathbf{E}_1\mathbf{A}$ by multiplying row 1 by 5:

$$\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 0.2 & -0.4 & 1 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{bmatrix} \longrightarrow \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{bmatrix}$$

Question L32.2: How is $\det(\mathbf{A})$ related to $\det(\mathbf{E}_2\mathbf{E}_1\mathbf{A})$?

$\det(\mathbf{A}) = -\frac{1}{5}\det(\mathbf{E}_2\mathbf{E}_1\mathbf{A})$. We need to divide by 5 here to compensate for factor of 5 by which the determinant increased.

An example of pivotal condensation, continued

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & -0.4 & 1 \\ -1 & 4 & 9 \end{bmatrix}$ $\quad \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{bmatrix}$

Then $\det(\mathbf{A}) = -\frac{1}{5}\det(\mathbf{E}_2\mathbf{E}_1\mathbf{A})$.

Form $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}$ by adding row 1 to row 3:

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{bmatrix} \longrightarrow \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 14 \end{bmatrix}$$

Question L32.3: How is $\det(\mathbf{A})$ related to $\det(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A})$?

The determinant does not change under this operation, so that:

$\det(\mathbf{A}) = -\frac{1}{5}\det(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = -\frac{1}{5}\det(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A})$.

An example of pivotal condensation, continued

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & -0.4 & 1 \\ -1 & 4 & 9 \end{bmatrix}$ $\quad \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 14 \end{bmatrix}$

Then $\det(\mathbf{A}) = -\frac{1}{5}\det(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A})$.

Form $\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}$ by subtracting 2(row 2) from row 3:

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 14 \end{bmatrix} \longrightarrow \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{bmatrix} = \mathbf{U}$$

Question L32.4: How is $\det(\mathbf{A})$ related to $\det(\mathbf{U})$?

The determinant does not change under this operation, so that:

$$\det(\mathbf{A}) = -\frac{1}{5}\det(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = -\frac{1}{5}\det(\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = -\frac{1}{5}\det(\mathbf{U}).$$

An example of pivotal condensation, completed

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & -0.4 & 1 \\ -1 & 4 & 9 \end{bmatrix}$ $\quad \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{bmatrix} = \mathbf{U}$

Then $\det(\mathbf{A}) = -\frac{1}{5}\det(\mathbf{U})$.

Question L32.5: What is $\det(\mathbf{A})$?

The matrix $\mathbf{U}$ is upper triangular.

Its determinant is the product of the diagonal elements:

$\det(\mathbf{U}) = (1)(1)(10) = 10$.

We conclude that $\det(\mathbf{A}) = -\frac{1}{5}\det(\mathbf{U}) = -2$.

A more convenient notation for keeping track

In our example, the notation was chosen so that it explicitly relates to the definition. In practice, you may prefer the following notation for keeping track of how the determinant does or does not change at every step of pivotal condensation:

$$det(\mathbf{A}) = \begin{vmatrix} 0 & 1 & 2 \\ 0.2 & -0.4 & 1 \\ -1 & 4 & 9 \end{vmatrix} \overset{R1 \leftrightarrow R2}{=} - \begin{vmatrix} 0.2 & -0.4 & 1 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{vmatrix}$$

$$\det(\mathbf{A}) \overset{R1 \mapsto 5R1}{=} -\frac{\mathbf{1}}{\mathbf{5}} \begin{vmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ -1 & 4 & 9 \end{vmatrix} \overset{R3 \mapsto R3 + R1}{=} -\frac{\mathbf{1}}{\mathbf{5}} \begin{vmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 14 \end{vmatrix}$$

$$\det(\mathbf{A}) \overset{R3 \mapsto R3 - 2R2}{=} -\frac{\mathbf{1}}{\mathbf{5}} \begin{vmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{vmatrix} = -\frac{\mathbf{1}}{\mathbf{5}}(1)(1)(10) = -2.$$

Take-home message

We have defined $\det(\mathbf{A})$ in terms of how the value of this number behaves under elementary row operations (slide 4). These properties allow us to calculate $\mathbf{A}$ by the method of *pivotal condensation.*

This method is outlined on slide 4, and slides 5–10 contain a worked-out example that gives a template for using it.

Pivotal condensation works by performing enough steps of Gaussian elimination to reduce the given matrix to an upper-triangular one and keeping track of how the determinant changes at every step.