Lecture 33: More Properties of Determinants

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MATH3200: Applied Linear Algebra

Properties of determinants that we have already seen

Suppose $\mathbf{A}$ is any square matrix. In Lecture 31 we discussed the following properties of determinants:

1. When $\mathbf{B}$ is obtained by switching two rows of $\mathbf{A}$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.
2. When $\mathbf{B}$ is obtained by multiplying of one row of $\mathbf{A}$ by a scalar λ, then $\det(\mathbf{B}) = \lambda \det(\mathbf{A})$.
3. When $\mathbf{B}$ is obtained by adding a scalar multiple of one row of $\mathbf{A}$ to another row, then $\det(\mathbf{B}) = \det(\mathbf{A})$.
4. If $\mathbf{A}$ is upper-triangular or lower-triangular, then $\det(\mathbf{A})$ is the product of the diagonal elements.

Here we will show that determinants has a number of additional properties that make them very useful tools in many applications.

Behavior of determinants under elementary column operations

The behavior of determinants with respect to operations on the columns is analogous to the behavior with respect to operations on the rows:

Theorem

Suppose $\mathbf{A}$ *is any square matrix. Then*

1. *When* $\mathbf{B}$ *is obtained by switching two columns of* $\mathbf{A}$*, then* $\det(\mathbf{B}) = -\det(\mathbf{A})$.
2. *When* $\mathbf{B}$ *is obtained by multiplying of one column of* $\mathbf{A}$ *by a scalar* λ*, then* $\det(\mathbf{B}) = \lambda \det(\mathbf{A})$.
3. *When* $\mathbf{B}$ *is obtained by adding a scalar multiple of one column of* $\mathbf{A}$ *to another column, then* $\det(\mathbf{B}) = \det(\mathbf{A})$.

Recall that you have proved this theorem for the special case of 2×2 matrices already in Module 61.

Some Examples

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 0 & -2 & -1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 7 & 6 \\ 0 & -2 & -1 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} 2 & 7 & 3 \\ 5 & 7 & 6 \\ -2 & 0 & -1 \end{bmatrix}$

It can be shown that $\det(\mathbf{A}) = 3$.

Question L33.1: What is $\det(\mathbf{B})$?

Here **B** is obtained from **A** by adding two times column 1 to column 2.

This operation does not change the determinant.

Thus $\det(\mathbf{B}) = \det(\mathbf{A}) = 3$.

Question L33.2: What is $\det(\mathbf{C})$?

Here **C** can be obtained from **A** by first multiplying column 1 by 7, which changes the determinant by a factor of 7, and then switching columns 1 and 2, which flips the sign of the determinant.

Thus $\det(\mathbf{C}) = (-1)(7)\det(\mathbf{A}) = -21$.

Examples and observations: Rank *vs.* determinant

Consider $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

$$\det(\mathbf{I}_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1)(1) - (0)(0) = 1.$$

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = (1)(3) - (2)(4) = -5.$$

Question L33.3: What is $\det(\mathbf{B})$?

$$\det(\mathbf{B}) = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = (1)(8) - (2)(4) = 0.$$

Note that in the first two examples, the matrices have full rank, and their determinants are nonzero.

In the third example, the second row is a scalar multiple of the first. The matrix does not have full rank, and its determinant is zero.

Singular *vs.* nonsingular matrices

Definition

A square matrix $\mathbf{A}$ is *singular* if $\det(\mathbf{A}) = 0$.

A square matrix $\mathbf{A}$ is *non-singular* if $\det(\mathbf{A}) \neq 0$.

On the previous slide, you saw two non-singular matrices of full rank and one singular square matrix that did not have full rank. This observation generalizes:

Theorem

An $n \times n$ matrix $\mathbf{A}$ is singular if, and only if, $r(\mathbf{A}) < n$.

In Module 63 we will prove this result for the case of 2×2 matrices.

The above theorem allows on to determine, just based on the value $\det(\mathbf{A})$, whether a given square matrix has full rank and a number of other equivalent properties.

Singular matrices of order $n \times n$

Definition

An $n \times n$ matrix $\mathbf{A}$ is *singular* if $\det(\mathbf{A}) = 0$.

We can now expand the main theorem of Chapter 3:

Theorem

The following properties of an $n \times n$ matrix $\mathbf{A}$ are equivalent:

1. $\det(\mathbf{A}) = 0$; *that is, $\mathbf{A}$ is singular.*
2. $r(\mathbf{A}) < n$.
3. *$\mathbf{A}$ is not invertible, that is, $\mathbf{A}^{-1}$ does not exist.*
4. *The system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is underdetermined.*
5. *Each system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is either underdetermined or inconsistent.*
6. *The range of $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ is not all of $\mathbb{R}^n$.*
7. *The function $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ is not one-to-one.*

Non-singular matrices of order $n \times n$

We can also rephrase these results for the case when $\det(\mathbf{A})$ takes any value other than 0:

Theorem

The following properties of an $n \times n$ matrix $\mathbf{A}$ are equivalent:

1. $\mathbf{A}$ *is non-singular, that is,* $\det(\mathbf{A}) \neq 0$.
2. $\mathbf{A}$ *has full rank, that is,* $r(\mathbf{A}) = n$.
3. $\mathbf{A}$ *is invertible, that is,* $\mathbf{A}^{-1}$ *exists.*
4. *The system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has a unique solution.*
5. *Each system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has a unique solution.*
6. *The range of* $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ *is* $\mathbb{R}^n$.
7. *The function* $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ *is one-to-one.*

Determinants of products and transposes: Examples

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$

Then $\mathbf{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad \mathbf{AB} = \begin{bmatrix} -5 & 1 \\ -11 & 3 \end{bmatrix}$

$\det(\mathbf{A}) = (1)(4) - (2)(3) = -2.$

$\det(\mathbf{B}) = (-1)(0) - (1)(-2) = 2.$

Question L33.4: What is $\det(\mathbf{A}^T)$?

$\det(\mathbf{A}^T) = (1)(4) - (3)(2) = -2 = \det(\mathbf{A}).$

Question L33.5: What is $\det(\mathbf{AB})$?

$\det(\mathbf{AB}) = (-5)(3) - (1)(-11) = -4 = \det(\mathbf{A})\det(\mathbf{B}).$

Determinants of products and transposes: A theorem

The examples on the previous slide illustrate the following result:

Theorem

Let $\mathbf{A}, \mathbf{B}$ *be square matrices. Then*

(i) $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

(ii) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Now consider the identity matrix $\mathbf{I}$ of order $n \times n$. It is diagonal, hence upper triangular, and all elements on the diagonal are equal to 1. It follows that $\det(\mathbf{I}) = 1$.

Thus for $\mathbf{A}$ as in the theorem and $\mathbf{B} = \mathbf{A}^{-1}$ must have
$1 = \det \mathbf{I} = \det(\mathbf{AA}^{-1}) = \det(\mathbf{A})\det(\mathbf{A}^{-1})$.

Question L33.6: What can we deduce from this?

Determinants of inverse matrices

Theorem

Let $\mathbf{A}$ *be matrices of order* $n \times n$. *Then*

(i) If $\mathbf{A}$ *has an inverse matrix* $\mathbf{A}^{-1}$, *then* $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.

(ii) If $\det(\mathbf{A}) = 0$, *then* $\mathbf{A}^{-1}$ *does not exist;* $\mathbf{A}$ *is non-invertible.*

(iii) If $\det(\mathbf{A}) \neq 0$, *then* $\mathbf{A}^{-1}$ *exists;* $\mathbf{A}$ *is invertible.*

From the equations $1 = \det \mathbf{I} = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A}) \det(\mathbf{A}^{-1})$ we can directly deduce part (i) and also part (ii), because the product $\det(\mathbf{A}) \det(\mathbf{A}^{-1})$ can never be equal to 1 if $\det(\mathbf{A}) = 0$.

The third part follows from the observation that the procedure of pivotal condensation for calculating $\det(\mathbf{A})$ is essentially the same process that we used for finding the rank $r(\mathbf{A})$. When $\det(\mathbf{A}) \neq 0$, then we must end up with an upper-triangular matrix that has no zeros on its (main) diagonal, so that every one of its n columns will be pivotal and $r(\mathbf{A}) = n$, which implies $\mathbf{A}^{-1}$ exists.

Take-home message

Determinants behave with respect to elementary column operations in analogous ways as they behave with respect to elementary row operations.

We have $\det(\mathbf{A}^T) = \det(\mathbf{A})$ and $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

When $\det(\mathbf{A}) = 0$ the matrix $\mathbf{A}$ is *singular* and $\mathbf{A}^{-1}$ does not exist.

When $\det(\mathbf{A}) \neq 0$ the matrix $\mathbf{A}$ is *non-singular*, $\mathbf{A}^{-1}$ exists, and $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.

The fact whether or not a matrix is singular tells us a lot about its properties (sides 7, 8).