Lecture 34: Applications of Determinants to the Geometry of Linear Transformations

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Review linear transformations $L_{\mathbf{A}}$

Recall from Chapter 3 that when $\mathbf{A}$ is a 2×2 matrix, then $\mathbf{A}$ defines a linear transformation $L_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$L_{\mathbf{A}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

Similarly, when $\mathbf{A}$ is a 3×3 matrix, then $\mathbf{A}$ defines a linear transformation $L_{\mathbf{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$L_{\mathbf{A}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \mathbf{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Determinants and linear transformations: Example 1

Let α be an angle, and let $\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Recall from Chapter 3 that $L_{\mathbf{R}_\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by an angle of α. The transformation can be implemented by a sheet of paper without turning it over. It *preserves orientation* and *preserves the areas* of rectangular and other regions.

Question L34.1: What is $\det(\mathbf{R}_\alpha)$?

$\det(\mathbf{R}_\alpha) = \cos^2\alpha + \sin^2\alpha = 1.$

Determinants and linear transformations: Example 2

Let $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$

Recall that $L_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$ corresponds to a stretch by a factor of 3 in the x-direction and a compression by a factor of 2 in the y-direction. The transformation can be implemented by deforming a sheet of elastic material that lies flat on a surface without turning it over. It *preserves orientation.*

The rectangle with endpoints $(0,0),(1,0),(0,1),(1,1)$ of area 1 will be mapped to the rectangle with endpoints $(0,0),(3,0),(0,0.5),(3,0.5)$ of area 1.5.

One can show that this transformation *increases the areas* of any rectangle, or any other region, *by a factor of 1.5.*

Note that here $\det(\mathbf{A}) = 3(0.5) = 1.5$.

Determinants and linear transformations: Example 3

Let $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Recall that $L_{\mathbf{B}} : \mathbb{R}^2 \to \mathbb{R}^2$ switches the x- and y-coordinates. The transformation can be implemented by flipping over a sheet of paper along the diagonal. It *reverses orientation* but *preserves the areas* of all rectangles and other regions.

Note that here $\det(\mathbf{B}) = (0)(0) - (1)(1) = -1$.

Determinants and linear transformations: Example 4

Let $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$

Then $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x + 2y \end{bmatrix}$

It follows that $L_{\mathbf{C}} : \mathbb{R}^2 \to \mathbb{R}^2$ maps the entire x-y-plane onto the y-axis. In particular, each region will be mapped to a subset of a line, which has area 0.

Note that here $\det(\mathbf{C}) = (0)(2) - (0)(1) = 0$.

Question 34.2: Let $\mathbf{A}$ be a 2×2 matrix. How is $\det(\mathbf{A})$ related to geometric properties of the linear transformation $L_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$? Make a guess based on what you observed in the preceding examples.

The following theorem describes the pattern that we observed:

Theorem

Let $\mathbf{A}$ *be a matrix of order* 2×2 *and let* $L_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$ *be the transformation of the Euclidean plane that is defined by* $\mathbf{A}$. *Then*

(i) $L_{\mathbf{A}}$ *maps any region of area* A *onto a region of area* $|\det(\mathbf{A})|A$.

(ii) If $\det(\mathbf{A}) > 0$, *then* $L_{\mathbf{A}}$ *preserves orientation.*

(iii) If $\det(\mathbf{A}) < 0$, *then* $L_{\mathbf{A}}$ *reverses orientation.*

(iv) If $\det(\mathbf{A}) = 0$, *then* $L_{\mathbf{A}}$ *maps* $\mathbb{R}^2$ *to a lower-dimensional subspace.*

A similar pattern holds when $\mathbf{A}$ has order 3×3

Theorem

Let $\mathbf{A}$ *be a matrix of order* 3×3 *and let* $L_{\mathbf{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ *be the transformation of the Euclidean space that is defined by* $\mathbf{A}$*. Then*

(i) $L_{\mathbf{A}}$ *maps any region of volume* V *onto a region of volume* $|\det(\mathbf{A})|V$*.*

(ii) If $\det(\mathbf{A}) > 0$*, then* $L_{\mathbf{A}}$ *preserves orientation.*

(iii) If $\det(\mathbf{A}) < 0$*, then* $L_{\mathbf{A}}$ *reverses orientation.*

(iv) If $\det(\mathbf{A}) = 0$*, then* $L_{\mathbf{A}}$ *maps* $\mathbb{R}^3$ *to a lower-dimensional subspace.*

Here the transformation could be implemented as a continuous deformation and/or rotation of solid objects if, and only if, it preserves orientation.