Lecture 35: Calculating Determinants by Cofactor Expansion

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MATH3200: Applied Linear Algebra

The goal of this lecture

In this lecture you will learn an alternative method for calculating the determinant of a square matrix.

This method often works well when the matrix is *sparse*, that is, when most of its elements are equal to 0.

First we need to introduce two new concepts: *minors* and *cofactors.*

Minors

A *minor* of a matrix **A** is a **determinant** of any **square** submatrix of **A**.

Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$$

All of the following numbers are examples of minors:

$$\begin{vmatrix} 1 & 3 & 4 \\ 6 & 8 & 9 \\ 11 & 13 & 14 \end{vmatrix} = 0 \qquad \begin{vmatrix} 7 & 10 \\ 17 & 20 \end{vmatrix} = -30 \qquad \det([12]) = 12$$

Minors, continued

Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$$

Question L35.1: Is $\begin{bmatrix} 8 & 9 \\ 13 & 14 \end{bmatrix}$ a minor of **A**?

No. $\begin{bmatrix} 8 & 9 \\ 13 & 14 \end{bmatrix}$ is a square submatrix of **A**, but minors are

by definition numbers. The corresponding minor would be

its determinant $\begin{vmatrix} 8 & 9 \\ 13 & 14 \end{vmatrix} = -5$.

Cofactors

The *cofactor of the element a_{ij}* of a **square** matrix **A** is the product of $(-1)^{i+j}$ with the minor that is obtained by **removing the ith row and the jth column** of **A**.

Example: Find the cofactor of a_{23}:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \blacksquare & a_{14} \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ a_{31} & a_{32} & \blacksquare & a_{34} \\ a_{41} & a_{42} & \blacksquare & a_{44} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix} \longrightarrow \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$$\longrightarrow (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \longrightarrow (-1) \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

Cofactors: An example

The *cofactor of the element a_{ij}* of a **square** matrix **A** is the product of $(-1)^{i+j}$ with the minor that is obtained by **removing the i$^{\text{th}}$ row and the j$^{\text{th}}$ column** of **A**.

Question L35.2: Find the cofactor of a_{24} in $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 2 & 3 & 5 & 7 \\ 11 & 13 & 17 & 19 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ 2 & 3 & 5 & \blacksquare \\ 11 & 13 & 17 & \blacksquare \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 11 & 13 & 17 \end{bmatrix} \longrightarrow \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 11 & 13 & 17 \end{vmatrix}$$

$$\longrightarrow (-1)^{2+4} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 11 & 13 & 17 \end{vmatrix} \longrightarrow 1 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 11 & 13 & 17 \end{vmatrix} = 7$$

The method of expansion by cofactors

Let **A** be any square matrix.

We can calculate det(**A**) as follows:

1. Pick any row or column.
2. For each element of the chosen row or column, find its cofactor.
3. *Multiply each element* in the cosen row or column by its cofactor.
4. Sum the results.

The method works best if you choose the row or column along which you want to expand as one that contains a lot of zero elements.

Expansion by cofactors: An example

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 4 & 9 & 5 \end{bmatrix}$

Expand along the first column:

$$\det(\mathbf{A}) = 1(-1)^{1+1} \begin{vmatrix} 7 & 8 \\ 9 & 5 \end{vmatrix} + 6(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 9 & 5 \end{vmatrix} + 4(-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix} =$$

$$1(1)((7)(5) - (8)(9)) + 6(-1)((2)(5) - (3)(9)) + 4(1)((2)(8) - (3)(7))$$

$$\det(\mathbf{A}) = 1(-37) - 6(-17) + 4(-5) = 45.$$

Expansion by cofactors: 3×3 matrices

Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Expand along the first row:

$$\det(\mathbf{A}) = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
$$+ a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det(\mathbf{A}) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

$$\det(\mathbf{A}) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

We obtain the formula that was shown in a previous lecture.

Expansion by cofactors: Another example

Suppose you want to use cofactor expansion along a column of the

matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 6 & 7 & 8 & 9 \\ 4 & 0 & 0 & 5 \\ -2 & -1 & 0 & 1 \end{bmatrix}$ to compute $\det(\mathbf{A})$.

Question L35.3: Along which column should you expand?

Expand along the third column: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 6 & 7 & 8 & 9 \\ 4 & 0 & 0 & 5 \\ -2 & -1 & 0 & 1 \end{bmatrix}$

$$\det(\mathbf{A}) = 0 + 8(-1)^{2+3} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -2 & -1 & 1 \end{vmatrix} + 0 + 0 = 8(-1) \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -2 & -1 & 1 \end{vmatrix}$$

Expansion by cofactors: The example completed

$$\det(\mathbf{A}) = 0 + 8(-1)^{2+3} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -2 & -1 & 1 \end{vmatrix} + 0 + 0 = 8(-1) \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -2 & -1 & 1 \end{vmatrix}$$

Question L35.4: If we now want to expand along a row, which row should we choose?

Now expand along the second row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -2 & -1 & 1 \end{vmatrix} = 4(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} + 0 + 5(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -2 & -1 & 1 \end{vmatrix} = 4(-1)(5) + 5(-1)(3) = -35.$$

$$\det(\mathbf{A}) = 8(-1)(-35) = 280.$$

Summary

A *minor* of a matrix **A** is a **determinant** of any square submatrix of **A**.

The *cofactor of the element a_{ij}* of a **square** matrix **A** is the product of $(-1)^{i+j}$ with the minor that is obtained by **removing the i^{th} row and the j^{th} column** of **A**.

We can calculate det(**A**) by *cofactor expansion* as follows:

1. Pick any row or column.
2. For each element of the chosen row or column, find its cofactor.
3. Multiply each element in the chosen row or column by its cofactor.
4. Sum the results.

The method works best if you choose the row or column along which you expand as one that contains many zero elements.