Lecture 36: Eigenvectors and Eigenvalues: Introduction

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MATH3200: Applied Linear Algebra

Two motivating examples

Let $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad$ Then:

$$\mathbf{A}\vec{\mathbf{x}}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\vec{\mathbf{x}}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = 0.5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now consider $\mathbf{B} = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ Then:

$$\mathbf{B}\vec{\mathbf{x}}_3 = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{B}\vec{\mathbf{x}}_4 = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Question L36.1: What do the vectors $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \vec{\mathbf{x}}_4$ in the above examples have in common?

Eigenvectors and eigenvalues: Definition

Definition

A *nonzero* vector $\vec{\mathbf{x}}$ is an *eigenvector* (or *characteristic vector*) of a square matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Then λ is an *eigenvalue* (or *characteristic value*) of $\mathbf{A}$.

Note: An eigenvalue is allowed to be 0, but eigenvectors are not allowed to be zero vectors.

In our first motivating example:

$\mathbf{A}\vec{\mathbf{x}}_1 = 3\vec{\mathbf{x}}_1$, so that $\vec{\mathbf{x}}_1$ is an eigenvector of $\mathbf{A}$ with eigenvalue 3.

$\mathbf{A}\vec{\mathbf{x}}_2 = 0.5\vec{\mathbf{x}}_2$, so that $\vec{\mathbf{x}}_2$ is an eigenvector of $\mathbf{A}$ with eigenvalue 0.5.

In our second motivating example:

$\mathbf{B}\vec{\mathbf{x}}_3 = 5\vec{\mathbf{x}}_3$, so that $\vec{\mathbf{x}}_3$ is an eigenvector of $\mathbf{B}$ with eigenvalue 5.

$\mathbf{B}\vec{\mathbf{x}}_4 = 2\vec{\mathbf{x}}_4$, so that $\vec{\mathbf{x}}_4$ is an eigenvector of $\mathbf{B}$ with eigenvalue 2.

A third example

Let $\mathbf{C} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_5 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_6 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_7 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ Then:

$$\mathbf{C}\vec{\mathbf{x}}_5 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad \mathbf{C}\vec{\mathbf{x}}_6 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C}\vec{\mathbf{x}}_7 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Question L36.2: Is $\vec{\mathbf{x}}_5$ an eigenvector of $\mathbf{C}$?
If so, what is its eigenvalue?

Question L36.3: Is $\vec{\mathbf{x}}_6$ an eigenvector of $\mathbf{C}$?
If so, what is its eigenvalue?

Question L36.4: Is $\vec{\mathbf{x}}_7$ an eigenvector of $\mathbf{C}$?
If so, what is its eigenvalue?

A third example

Let $\mathbf{C} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_5 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_6 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_7 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ Then:

$$\mathbf{C}\vec{\mathbf{x}}_5 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad \mathbf{C}\vec{\mathbf{x}}_6 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C}\vec{\mathbf{x}}_7 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Answer L36.2: $\vec{\mathbf{x}}_5$ an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda = -2$.

Answer L36.3: $\vec{\mathbf{x}}_6$ is not an eigenvector of $\mathbf{C}$, because a zero vector is never an eigenvector.

Answer L36.4: $\vec{\mathbf{x}}_7$ an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda = 2$.

A fourth example

Let $\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_8 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_9 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_{10} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \vec{\mathbf{x}}_{11} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathbf{D}\vec{\mathbf{x}}_8 = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix} \quad \mathbf{D}\vec{\mathbf{x}}_9 = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

$$\mathbf{D}\vec{\mathbf{x}}_{10} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \quad \mathbf{D}\vec{\mathbf{x}}_{11} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Question L36.5: Is $\vec{\mathbf{x}}_8$ an eigenvector of $\mathbf{D}$?
If so, what is its eigenvalue?

Question L36.6: Is $\vec{\mathbf{x}}_9$ an eigenvector of $\mathbf{D}$?
If so, what is its eigenvalue?

Question L36.7: Is $\vec{\mathbf{x}}_{10}$ an eigenvector of $\mathbf{D}$?
If so, what is its eigenvalue?

Question L36.8: Is $\vec{\mathbf{x}}_{11}$ an eigenvector of $\mathbf{D}$?
If so, what is its eigenvalue?

A fourth example

Let $\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$ $\vec{\mathbf{x}}_8 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\vec{\mathbf{x}}_9 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $\vec{\mathbf{x}}_{10} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{\mathbf{x}}_{11} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathbf{D}\vec{\mathbf{x}}_8 = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix} \quad \mathbf{D}\vec{\mathbf{x}}_9 = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

$$\mathbf{D}\vec{\mathbf{x}}_{10} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \qquad \mathbf{D}\vec{\mathbf{x}}_{11} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Answer L36.5: $\vec{\mathbf{x}}_8$ an eigenvector of $\mathbf{D}$ with eigenvalue $\lambda = 4$.

Answer L36.6: $\vec{\mathbf{x}}_9$ an eigenvector of $\mathbf{D}$ with eigenvalue $\lambda = 4$.

Answer L36.7: $\vec{\mathbf{x}}_{10}$ is not an eigenvector of $\mathbf{D}$.

Answer L36.8: $\vec{\mathbf{x}}_{11}$ is an eigenvector of $\mathbf{D}$ with eigenvalue $\lambda = 0$.

An important observation

Proposition

Let $\mathbf{A}$ *be a square matrix.*

If $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ*, then for every scalar* $c \neq 0$ *the vector* $c\vec{\mathbf{x}}$ *is also an eigenvector of* $\mathbf{A}$ *with the same eigenvalue* λ*.*

Proof: Let $\vec{\mathbf{x}}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ, and let $c \neq 0$.

We need to show that $c\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ and $\mathbf{A}(c\vec{\mathbf{x}}) = \lambda(c\vec{\mathbf{x}})$.

The former follows from the assumptions that $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ and $c \neq 0$.

By properties of matrix multiplication and our assumptions:

$\mathbf{A}(c\vec{\mathbf{x}}) = c(\mathbf{A}\vec{\mathbf{x}}) = c(\lambda\vec{\mathbf{x}}) = \lambda(c\vec{\mathbf{x}})$. $\square$

Another important observation

Proposition

Let $\mathbf{A}$ *be a square matrix.*

The matrix $\mathbf{A}$ *has an eigenvalue* $\lambda = 0$ *if, and only if,* $\mathbf{A}$ *is singular.*

Proof: Note that $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 0 if, and only if, $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ and $\mathbf{A}\vec{\mathbf{x}} = 0\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

This means that the homogeneous system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a nonzero solution and is underdetermined (as $\vec{\mathbf{0}}$ is always a solution).

By the theorem at the end of Conversation 30, the latter property is equivalent to $\det(\mathbf{A}) = 0$, which means that $\mathbf{A}$ is singular.

Example 1 revisited: Some observations

Let $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$

- $\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 3$.
- $\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 0.5$.
- The eigenvalues are distinct, $\lambda_1 \neq \lambda_2$.
- These two eigenvectors are linearly independent.

Example 2 revisited: Some observations

Let $\mathbf{B} = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$

- $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 5$.
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 2$.
- The eigenvalues are distinct, $\lambda_1 \neq \lambda_2$.
- These two eigenvectors are linearly independent.

Example 3 revisited: Some observations

Let $\mathbf{C} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$

- $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -2$.

- $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 2$.

- The eigenvalues are distinct, $\lambda_1 \neq \lambda_2$.
- These two eigenvectors are linearly independent.

Example 4 revisited: Some observations

Let $\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$

- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 0$.
- $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 4$.
- The eigenvalues are distinct, $\lambda_1 \neq \lambda_2$.
- These two eigenvectors are linearly independent.

These observations generalize

Definition

Let **A** be an $n \times n$ matrix. We say that **A** has a *full set of eigenvectors* if there exist n eigenvectors of **A** that form a linearly independent set.

Theorem

Let **A** *be an* $n \times n$ *matrix. Assume* **A** *has* n *pairwise distinct real eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Then **A** *has a full set of eigenvectors.*

We will see later why we need to explicitly assume here that all eigenvalues are real numbers.

Eigenvalues and eigenvectors: The easy case

Suppose $\mathbf{D}$ is a diagonal matrix.

Consider a standard basic vector $\vec{\mathbf{e}}_i$. Then

$$\mathbf{D}\vec{\mathbf{e}}_i = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_i & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} = \lambda_i \vec{\mathbf{e}}_i.$$

Thus $\vec{\mathbf{e}}_i$ is an eigenvector with eigenvalue λ_i.

The set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \dots, \vec{\mathbf{e}}_n\}$ is linearly independent.

Thus $\mathbf{D}$ has a full set of eigenvectors, regardless of whether or not the eigenvalues are all distinct.

Take-home message

A *nonzero* vector $\vec{\mathbf{x}}$ is an *eigenvector* of a square matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Then λ is an *eigenvalue* of $\mathbf{A}$.

An eigenvalue is allowed to be 0, but eigenvectors are not allowed to be zero vectors.

If $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then for every scalar $c \neq 0$ the vector $c\vec{\mathbf{x}}$ is also an eigenvector of $\mathbf{A}$ with the same eigenvalue λ.

The matrix $\mathbf{A}$ has an eigenvalue $\lambda = 0$ if, and only if, $\mathbf{A}$ is singular.

A square matrix $\mathbf{A}$ has a *full set of eigenvectors* if there exist n eigenvectors of $\mathbf{A}$ that form a linearly independent set.

If $\mathbf{A}$ has order $n \times n$ and has n pairwise distinct eigenvalues, then $\mathbf{A}$ has a full set of eigenvectors.

For a diagonal matrix $\mathbf{D}$, the eigenvalues are the elements of the (main) diagonal, and the eigenvectors are the standard basis vectors $\vec{\mathbf{e}}_i$ that form a full set of eigenvectors of $\mathbf{D}$.