Lecture 37A: Finding Eigenvalues

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MATH3200: Applied Linear Algebra

The goal of this lecture

In the previous lecture we introduced the notions of eigenvectors and eigenvalues.

In this lecture and the next we will learn **how to find** the eigenvectors and eigenvalues of a given square matrix $\mathbf{A}$.

This involves a two-stage procedure:

1. First we find all the eigenvalues of $\mathbf{A}$.
2. Then we find—for each eigenvalue λ—the eigenvectors with this eigenvalue.

This lecture focuses on the first stage of this procedure; the next lecture will focus on the second step.

Review: Eigenvectors and eigenvalues

Definition

A vector $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ is an *eigenvector* of a square matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Then λ is an *eigenvalue* of $\mathbf{A}$.

Recall that a zero vector $\vec{\mathbf{0}}$ can never be an eigenvector, but the number $\lambda = 0$ can be an eigenvalue of a square matrix.

Let us begin by closely studying this definition and making a few observations. These will naturally lead us to a step-by-step procedure for finding the eigenvalues of a given square matrix $\mathbf{A}$.

An observation about these definitions

A scalar λ is an eigenvalue of a given square matrix $\mathbf{A}$ if, and only if, there exists a *nonzero* vector $\vec{\mathbf{x}} = [x_1, \dots, x_n]^T$ such that

$$\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (\lambda \mathbf{I})\vec{\mathbf{x}}$$

It follows that $\vec{\mathbf{x}}$ is an eigenvector with eigenvalue λ of a square matrix $\mathbf{A}$ if, and only if,

$\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ and $\mathbf{A}\vec{\mathbf{x}} = \lambda \mathbf{I}\vec{\mathbf{x}}$.

This version of the definition allows us to design a method for finding the eigenvalues.

When is λ an eigenvalue?

We observed that $\vec{\mathbf{x}}$ is an eigenvector with eigenvalue λ of a square matrix $\mathbf{A}$ if, and only if,

$\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ and $\mathbf{A}\vec{\mathbf{x}} = \lambda \mathbf{I}\vec{\mathbf{x}}$.

We can rewrite the last equation as:

$\mathbf{A}\vec{\mathbf{x}} - \lambda \mathbf{I}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

Use the Right Distributivity Law to factor out $\vec{\mathbf{x}}$:

$(\mathbf{A} - \lambda \mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

This is a homogeneous system of linear equations with coefficient matrix $(\mathbf{A} - \lambda \mathbf{I})$.

Question L37.1: When does this system have a solution $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$?

The system $(\mathbf{A} - \lambda \mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a solution $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ if, and only if, it is underdetermined.

This will be the case if, and only if, $\mathbf{A} - \lambda \mathbf{I}$ is a singular matrix, that is, a matrix such that $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.

The characteristic polynomial of **A**

We have proved the following result:

Theorem

Let **A** *be a square matrix. Then a scalar* λ *is an eigenvalue of* **A** *if, and only if,* $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

The expression $\det(\mathbf{A} - \lambda\mathbf{I})$ in the above theorem depends on λ; it is a function of λ.

More precisely, for an $n \times n$ matrix **A**, the expression $\det(\mathbf{A} - \lambda\mathbf{I})$ is a polynomial in λ of degree n, called the *characteristic polynomial of* **A**.

Question L37.2: How are the eigenvalues of **A** related to the characteristic polynomial of **A**, and how can we use the characteristic polynomial for finding the eigenvalues of **A**?

The eigenvalues of **A** are the roots of the characteristic polynomial of **A**. We can find them by factoring the characteristic polynomial.

The characteristic polynomial: Example 1

Consider $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$

Then we have:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 8-\lambda & 3 \\ -6 & -1-\lambda \end{bmatrix}$$

Thus $\det(\mathbf{A} - \lambda\mathbf{I}) = (8-\lambda)(-1-\lambda) - (3)(-6) = \lambda^2 - 7\lambda + 10.$

We can factor this quadratic polynomial as follows:

$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5).$

Question L37.3: What are the eigenvalues of $\mathbf{A}$?

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 5$.

The characteristic polynomial: Example 2

Consider $\mathbf{B} = \begin{bmatrix} 3 & -1 \\ 0 & 0.5 \end{bmatrix}$

Form $\mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix} 3 & -1 \\ 0 & 0.5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ 0 & 0.5-\lambda \end{bmatrix}$

Then $\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(0.5-\lambda) - (-1)(0) = (3-\lambda)(0.5-\lambda)$.

The eigenvalues are the elements $\lambda_1 = 3$ and $\lambda_2 = 0.5$ on the main diagonal.

This pattern generalizes

Consider any upper-triangular matrix $\mathbf{U} = \begin{bmatrix} \lambda_1 & u_{12} & \dots & u_{1n} \\ 0 & \lambda_2 & \dots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$

Form $\mathbf{U} - \lambda \mathbf{I} = \begin{bmatrix} \lambda_1 - \lambda & u_{12} & \dots & u_{1n} \\ 0 & \lambda_2 - \lambda & \dots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n - \lambda \end{bmatrix}$

Then $\det(\mathbf{U} - \lambda \mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$.
Here λ is a variable, while $\lambda_1, \lambda_2, \dots, \lambda_n$ are given constants.

The eigenvalues are the elements $\lambda_1, \lambda_2, \dots, \lambda_n$ on the (main) diagonal.

Note that some number λ_i may occur multiple times in this list. It is then called a *repeated* eigenvalues or an eigenvalue of *multiplicity* $k_i > 1$.

The characteristic polynomial: Example 3

Consider $\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 7 & 8 \\ 4 & 0 & 3 \end{bmatrix}$ Then $\mathbf{C} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 0 & 2 \\ 6 & 7-\lambda & 8 \\ 4 & 0 & 3-\lambda \end{bmatrix}$

Question L37.4: How should we calculate $\det(\mathbf{C} - \lambda\mathbf{I})$?

We can use cofactor expansion along the second column:

$$\begin{aligned} \det(\mathbf{C} - \lambda\mathbf{I}) &= 0 + (-1)^{2+2}(7-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} + 0 \\ &= (7-\lambda)((1-\lambda)(3-\lambda) - 8) \\ &= (7-\lambda)(\lambda^2 - 4\lambda - 5) = (7-\lambda)(\lambda-5)(\lambda+1) \end{aligned}$$

Question L37.5: What are the eigenvalues of $\mathbf{C}$?

The eigenvalues of $\mathbf{C}$ are $\lambda_1 = 7$, $\lambda_2 = 5$, and $\lambda_3 = -1$.

Summary

In this lecture we learned a method for finding all eigenvalues and eigenvectors of a given square matrix $\mathbf{A}$.

To do so, we first form the *characteristic polynomial* $\det(\mathbf{A} - \lambda\mathbf{I})$ of the matrix $\mathbf{A}$.

For a 2×2 matrix $\mathbf{A}$, the characteristic polynomial can be computed from the formula for the determinant. For square matrices of larger order one can use cofactor expansion.

The eigenvalues of $\mathbf{A}$ are then the roots of its characteristic polynomial and can be found by factoring it.

When $\mathbf{A}$ is an upper-triangular matrix or a lower-triangular matrix, then the eigenvalues of $\mathbf{A}$ are its elements on the (main) diagonal.