Lecture 37B: Finding Eigenvectors

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MATH3200: Applied Linear Algebra

The goal of this lecture

Recall that a vector $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ is an eigenvector of a square matrix $\mathbf{A}$ if there exists a scalar λ, called the eigenvalue of $\vec{\mathbf{x}}$, such that $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Finding eigenvectors and eigenvalues of a given square matrix $\mathbf{A}$ involves a two-stage procedure:

1. First we find all the eigenvalues of $\mathbf{A}$.
2. Then we find—for each eigenvalue λ—all eigenvectors with this eigenvalue.

In Lecture 37A we focused on the first stage; here we will focus on the second stage. Our first few examples will be the same as in Lecture 37A.

Finding eigenvectors of a given square matrix $\mathbf{A}$

1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$.
2. Factor the characteristic polynomial. The roots are the eigenvalues of $\mathbf{A}$.
3. For each (real) eigenvalue λ_i, find the eigenvectors $\vec{\mathbf{x}}$ with this eigenvalue as follows:
   1. Form $\mathbf{A} - \lambda_i\mathbf{I}$ by subtracting the number λ_i from each diagonal element of $\mathbf{A}$.
   2. Solve the system of linear equations $(\mathbf{A} - \lambda_i\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$, for example by Gaussian elimination.
   3. Your solution will contain at least 1 and up to k_i variables x_j that you can choose arbitrarily. Here k_i denotes the multiplicity of eigenvalue λ_i. For each of these variables x_j, find an eigenvector by setting it to 1, while setting the other variables that you can choose freely to 0.

This procedure will always give you a maximal linearly independent set of eigenvectors of $\mathbf{A}$. When all λ_i are pairwise distinct, it is guaranteed to give you a full set of eigenvectors.

Finding eigenvectors: Example 1 revisited

For $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$ we found the eigenvalues $\lambda_1 = 2, \lambda_2 = 5$.

Finding an eigenvector with eigenvalue $\lambda_1 = 2$:

Form $\mathbf{A} - \lambda_1 \mathbf{I} = \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 8-2 & 3 \\ -6 & -1-2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -6 & -3 \end{bmatrix}$

The system $(\mathbf{A} - 2\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ can be writted as $\begin{array}{rcl} 6x_1 + 3x_2 & = & 0 \\ -6x_1 - 3x_2 & = & 0 \end{array}$

It reduces to one equation. Each vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix}$ is a solution.

Setting $x_1 = 1$, we find that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an example of an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = 2$.

Finding eigenvectors: Example 1 completed

For $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$ we found the eigenvalues $\lambda_1 = 2, \lambda_2 = 5$.

Finding an eigenvector with eigenvalue $\lambda_2 = 5$:

Form $\mathbf{A} - \lambda_2\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 8-5 & 3 \\ -6 & -1-5 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -6 & -6 \end{bmatrix}$

The system $(\mathbf{A} - 5\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ can be written as $\begin{aligned} 3x_1 + 3x_2 &= 0 \\ -6x_1 - 6x_2 &= 0 \end{aligned}$

It reduces to one equation. Each vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ is a solution.

Question L37.6: Find one particular eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 5$.

Setting $x_1 = 1$, we find that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an example of an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 5$.

Finding eigenvectors: Example 2 revisited

For $\mathbf{B} = \begin{bmatrix} 3 & -1 \\ 0 & 0.5 \end{bmatrix}$ we found the eigenvalues $\lambda_1 = 3, \lambda_2 = 0.5$.

Finding an eigenvector with eigenvalue $\lambda_1 = 3$:

Form $\mathbf{B} - \lambda_1 \mathbf{I} = \mathbf{B} - 3\mathbf{I} = \begin{bmatrix} 3-3 & -1 \\ 0 & 0.5-3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -2.5 \end{bmatrix}$

The system $(\mathbf{B} - 3\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ can be writted as $\begin{array}{rcl} -x_2 & = & 0 \\ -2.5x_2 & = & 0 \end{array}$

Question L37.7: Find the set of all eigenvectors with eigenvalue $\lambda_1 = 3$.

Each vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ with $x_1 \neq 0$ is an eigenvector with eigenvalue 3.

Setting $x_1 = 1$, we can choose $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as one of these eigenvectors.

Finding eigenvectors: Example 2 completed

For $\mathbf{B} = \begin{bmatrix} 3 & -1 \\ 0 & 0.5 \end{bmatrix}$ we found the eigenvalues $\lambda_1 = 3, \lambda_2 = 0.5$.

Finding an eigenvector with eigenvalue $\lambda_2 = 0.5$:

Form $\mathbf{B} - \lambda_2 \mathbf{I} = \mathbf{B} - 0.5\mathbf{I} = \begin{bmatrix} 3 - 0.5 & -1 \\ 0 & 0.5 - 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 & -1 \\ 0 & 0 \end{bmatrix}$

Question 37.8: How can the system $(\mathbf{B} - 0.5\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ be written as two linear equations?

$$\begin{aligned} 2.5x_1 - x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

Each vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ 2.5x_1 \end{bmatrix}$ is a solution.

Setting $x_1 = 1$, we find that $\begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$ is an example of an eigenvector of $\mathbf{B}$ with eigenvalue $\lambda_2 = 0.5$.

Finding eigenvectors: Example 3 revisited

For $\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 7 & 8 \\ 4 & 0 & 3 \end{bmatrix}$ we found the eigenvalues $\lambda_1 = 7, \lambda_2 = 5, \lambda_3 = -1$.

Finding an eigenvector with eigenvalue $\lambda_1 = 7$:

Form $\mathbf{C} - \lambda_1 \mathbf{I} = \mathbf{C} - 7\mathbf{I} = \begin{bmatrix} 1-7 & 0 & 2 \\ 6 & 7-7 & 8 \\ 4 & 0 & 3-7 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 2 \\ 6 & 0 & 8 \\ 4 & 0 & -4 \end{bmatrix}$

Then $(\mathbf{C} - 7\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ can be writted as
$$\begin{array}{rcrcr} -6x_1 & + & 2x_3 & = & 0 \\ 6x_1 & + & 8x_3 & = & 0 \\ 4x_1 & - & 4x_3 & = & 0 \end{array}$$

All solutions must satisfy $x_1 = x_3 = 0$, but x_2 can be arbitrary.

Each vector $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$ with $x_2 \neq 0$ is an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda_1 = 7$.

Finding eigenvectors: Example 3 continued

For $\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 7 & 8 \\ 4 & 0 & 3 \end{bmatrix}$ we found the eigenvalues $\lambda_1 = 7, \lambda_2 = 5, \lambda_3 = -1$.

Finding an eigenvector with eigenvalue $\lambda_2 = 5$:

Form $\mathbf{C} - \lambda_2 \mathbf{I} = \mathbf{C} - 5\mathbf{I} = \begin{bmatrix} 1-5 & 0 & 2 \\ 6 & 7-5 & 8 \\ 4 & 0 & 3-5 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ 6 & -2 & 8 \\ 4 & 0 & -2 \end{bmatrix}$

The system $(\mathbf{A} - 5\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ can be writted as

$$\begin{array}{rcrcrcl} -4x_1 & & & + & 2x_3 & = & 0 \\ 6x_1 & - & 2x_2 & + & 8x_3 & = & 0 \\ 4x_1 & & & - & 2x_3 & = & 0 \end{array}$$

Now the eigenvectors with eigenvalue $\lambda_2 = 5$ can be found by solving this system by Gaussian elimination.

We will complete this example in Module 67B.

An example with repeated eigenvalues

Let $\mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Then $\lambda = 4$ is an eigenvalue of multiplicity $k = 2$ of $\mathbf{D}$.

Form $\mathbf{D} - \lambda\mathbf{I} = \mathbf{D} - 4\mathbf{I} = \begin{bmatrix} 4-4 & 0 \\ 0 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Every 2×1 column vector $\vec{\mathbf{x}}$ is a solution of the system

$$(\mathbf{D} - 4\mathbf{I})\vec{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{\mathbf{x}} = \vec{\mathbf{0}}.$$

Question L37.9: Does it follow that every 2×1 column vector $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{D}$?

No. It does follow though that every nonzero 2×1 column vector $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{D}$.

The example with repeated eigenvalues completed

Let $\mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Then every vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of $\mathbf{D}$.

When we choose $x_1 = 1$ and $x_2 = 0$, we find that

$\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\mathbf{D}$ with eigenvalue $\lambda = 4$.

Similarly, when we choose $x_2 = 1$ and $x_1 = 0$, we find that

$\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{D}$ with eigenvalue $\lambda = 4$.

Thus $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a maximal linearly independent

set of eigenvectors of $\mathbf{D}$ with eigenvalue $\lambda = 4$.

Summary

In this lecture and in Lecture 37A we learned a method for finding all eigenvalues and eigenvectors of a given square matrix $\mathbf{A}$.

To do so, we first compute the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$ of the matrix $\mathbf{A}$. For a 2×2 matrix, use the formula for the determinant; for square matrices of higher order, use pivotal condensation.

The eigenvalues of $\mathbf{A}$ are then the roots of its characteristic polynomial and can be found by factoring it.

When $\mathbf{A}$ is an upper-triangular matrix or a lower-triangular matrix, then the eigenvalues of $\mathbf{A}$ are its elements on the (main) diagonal.

For each eigenvalue λ, the eigenvectors of $\mathbf{A}$ with eigenvalue λ can then be found by solving the linear system $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
The procedure for this last stage is described in detail on slide 3.