Lecture 38: Eigenvectors and Eigenvalues of Inverse Matrices and Matrix Transposes

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MATH3200: Applied Linear Algebra

An example

In this lecture we will explore how the eigenvalues and eigenvectors of a square matrix $\mathbf{A}$ are related to the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$ and $\mathbf{A}^T$.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & -0.25 \end{bmatrix}$

Then $\mathbf{AB} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & -0.25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Thus $\mathbf{B} = \mathbf{A}^{-1}$.

In Lecture 36 we had explored the matrix $\mathbf{A}$ and found:

- $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = -2$.
- $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 2$.

An example, continued

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ Then $\mathbf{A}^{-1} = \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & -0.25 \end{bmatrix}$

$\vec{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = -2$.

Question L38.1: Is $\vec{\mathbf{x}}_1$ also an eigenvector of $\mathbf{A}^{-1}$?
If so, what is its eigenvalue?

$\vec{\mathbf{x}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 2$.

Question L38.2: Is $\vec{\mathbf{x}}_2$ also an eigenvector of $\mathbf{A}^{-1}$?
If so, what is its eigenvalue?

An example: Eigenvalues and eigenvectors of $\mathbf{A}^{-1}$

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ Then $\mathbf{A}^{-1} = \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & -0.25 \end{bmatrix}$

- $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = -2$.
- $\begin{bmatrix} 0.25 & 0.75 \\ 0.25 & -0.25 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda_1} = \frac{1}{-2}$.

- $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 2$.
- $\begin{bmatrix} 0.25 & 0.75 \\ 0.25 & -0.25 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda_2} = \frac{1}{2}$.

Eigenvalues and eigenvectors of inverse matrices

This example illustrates the following general result:

Theorem

Let $\mathbf{A}$ *be an invertible matrix.*

Let $\vec{\mathbf{x}}$ *be an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ*.*

Then $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}^{-1}$ *with eigenvalue* $\frac{1}{\lambda}$*.*

In other words, $\mathbf{A}$ and $\mathbf{A}^{-1}$ have the same eigenvectors, and the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

How about the eigenvectors of the transpose?

Let $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$ Then $\mathbf{A}^T = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$

in Module 67 we found that:

- $\vec{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = 2$.
- $\vec{\mathbf{x}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 5$.

Question L38.3: Is the vector $\vec{\mathbf{x}}_2$ above an eigenvector of $\mathbf{A}^T$? If so, what is its eigenvalue?

$$\mathbf{A}^T \vec{\mathbf{x}}_2 = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus $\vec{\mathbf{x}}_2$ is not an eigenvector of $\mathbf{A}$. Similarly, neither is $\vec{\mathbf{x}}_1$.

So how about the eigenvectors of the transpose?

Let $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$ Then $\mathbf{A}^T = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$

in Module 67 and Lecture 36 we found that:

- $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = 2$.
- $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 5$.
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}^T$ with eigenvalue $\lambda_2 = 2$.
- $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}^T$ with eigenvalue $\lambda_1 = 5$.

Thus in this example, the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^T$ are the same, but the eigenvectors of these matrices are different.

A theorem about the eigenvalues of the transpose

The preceding example illustrates the following general theorem.

Theorem

Let $\mathbf{A}$ *be a square matrix, and let* λ *be an eigenvalue of* $\mathbf{A}$*.*

Then λ *is an eigenvalue of* $\mathbf{A}^T$*.*

Thus a square matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$ always have the same eigenvalues, but not necessarily the same eigenvectors.

In fact, there is no obvious relationship between the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^T$.

However, it turns out that the transposes of the eigenvectors of $\mathbf{A}$ will always be so-called *left eigenvectors* of $\mathbf{A}^T$ and *vice versa.*

(Right) eigenvectors and left eigenvectors

Recall that a *column* vector $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ is an *eigenvector* with eigenvalue λ of a square matrix $\mathbf{A}$ if $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Eigenvectors are sometimes also called a *right eigenvectors.*

Definition

A *row* vector $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ is a *left eigenvector* with eigenvalue λ of a square matrix $\mathbf{A}$ if $\vec{\mathbf{x}}\mathbf{A} = \lambda\vec{\mathbf{x}}$.

For example, $[1, -1]\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = [-1, 1] = (-1)[1, -1]$.

Thus $[1, -1]$ is a left eigenvector of $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

with eigenvalue $\lambda = -1$.

A theorem about left eigenvectors of $\mathbf{A}^T$

Now suppose $\vec{\mathbf{x}}$ is an eigenvector with eigenvalue λ of $\mathbf{A}$.

This means that $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ and $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Then $\vec{\mathbf{x}}^T \neq \vec{\mathbf{0}}$ and $(\mathbf{A}\vec{\mathbf{x}})^T = \vec{\mathbf{x}}^T\mathbf{A}^T = (\lambda\vec{\mathbf{x}})^T = \lambda\vec{\mathbf{x}}^T$.

Thus $\vec{\mathbf{x}}^T$ is a left eigenvector of $\mathbf{A}^T$ with the same eigenvalue λ.

This observation proves the following result:

Theorem

Let $\mathbf{A}$ be a square matrix, and let $\vec{\mathbf{x}}$ be an eigenvector with eigenvalue λ of $\mathbf{A}$.

Then $\vec{\mathbf{x}}^T$ is a left eigenvector of $\mathbf{A}$ with eigenvalue λ.

Similarly, if $\vec{\mathbf{y}}$ is a left eigenvector of $\mathbf{A}$ with eigenvalue λ, then $\vec{\mathbf{y}}^T$ is an eigenvector with eigenvalue λ of $\mathbf{A}^T$.

How about left eigenvectors of the transpose?

Let $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$ Then $\mathbf{A}^T = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$

in Module 67 we found that:

- $\vec{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = 2$.
- $\vec{\mathbf{x}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 5$.

Question L38.4: Find a left eigenvector of $\mathbf{A}^T$ with eigenvalue $\lambda = 2$.

$$\vec{\mathbf{x}}_1^T \mathbf{A}^T = [1, -2] \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} = [2, -4] = 2\,[1, -2].$$

Thus $\vec{\mathbf{x}}_1^T = [1, -2]$ is a left eigenvector of $\mathbf{A}$ with eigenvalue $\lambda = 2$.

Take-home message

Theorem

Let $\mathbf{A}$ be an invertible matrix, and let $\vec{\mathbf{x}}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ.

Then $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda}$.

In other words, $\mathbf{A}$ and $\mathbf{A}^{-1}$ have the same eigenvectors, and the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

Theorem

Let $\mathbf{A}$ be a square matrix. Then $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues.

However, $\mathbf{A}$ and $\mathbf{A}^T$ usually have different eigenvectors.

But if $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then $\vec{\mathbf{x}}^T$ is a left eigenvector with eigenvalue λ of $\mathbf{A}^T$, which means that $\vec{\mathbf{x}}^T\mathbf{A}^T = \lambda\vec{\mathbf{x}}^T$.

Similarly, if $\vec{\mathbf{y}}$ is a left eigenvector of $\mathbf{A}$ with eigenvalue λ, then $\vec{\mathbf{y}}^T$ is an eigenvector with eigenvalue λ of $\mathbf{A}^T$.