Lecture 3: More Examples of Matrices: Adjacency Matrices of Graphs

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MATH3200: Applied Linear Algebra

A graph of friendships

Assume your instructor is nosy and wants to know about the friendships formed among the students in the class.

This information can be graphically represented by a *graph G:*

Draw a little circle for each student i.
These circles represent the *nodes* or *vertices* of G.

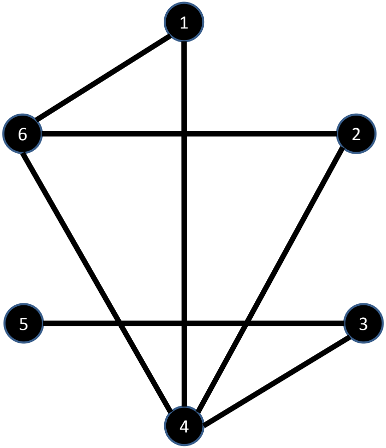
Then connect nodes i and j with a line segment if, and only if, i and j are friends.
These line segments represent the *edges* of G.

We are assuming here that i doesn't count as friend of i him- or herself, so that there are no loops in G from i to i.
This makes the grah G *loop-free.*

Also, we draw at most one line segment between any two given nodes. This makes the graph G *simple.*

We also assume that friendship is always reciprocated, so that when j is a friend of i, then also i is a friend of j.
This makes the graph G *undirected.*

A graph of friendships



A matrix of friendships

Consider the graph of friendships on the previous slide.

How can we represent this graph as a matrix?

We can represent a graph G with n vertices in the form of its *adjacency matrix* $\mathbf{A} = [a_{ij}]_{n \times n}$,

where $a_{ij} = 1$ when there is an edge between i and j in G

and $a_{ij} = 0$ otherwise.

In our example, there are $n = 6$ nodes, so that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Finding the first entry

Question L3.1: What is a_{11}?

There is no edge from node 1 to node 1 so that $a_{11} = 0$:

$$\mathbf{A} = \begin{bmatrix} {\color{red}0} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

The (main) diagonal of **A**

In fact, since our graph is simple, all the elements a_{ii} on the *(main) diagonal* of **A** will be zeros:

$$\mathbf{A} = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 0 \end{bmatrix}$$

Finding an off-diagonal element

Question L3.2: What is a_{14}?

There is an edge between nodes 1 and 4 so that $a_{14} = 1$:

$$\mathbf{A} = \begin{bmatrix} 0 & a_{12} & a_{13} & \color{red}{1} & a_{15} & a_{16} \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 0 \end{bmatrix}$$

For the edges of G, we put ones into $\mathbf{A}$

The same edge also requires $a_{41} = 1$:

$$\mathbf{A} = \begin{bmatrix} 0 & a_{12} & a_{13} & 1 & a_{15} & a_{16} \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} & a_{36} \\ 1 & a_{42} & a_{43} & 0 & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 0 \end{bmatrix}$$

There is an edge between nodes 1 and 6 so that $a_{16} = 1 = a_{61}$:

$$\mathbf{A} = \begin{bmatrix} 0 & a_{12} & a_{13} & 1 & a_{15} & 1 \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} & a_{36} \\ 1 & a_{42} & a_{43} & 0 & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 & a_{56} \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & 0 \end{bmatrix}$$

When there is no edge in G, we put zero into $\mathbf{A}$

There are no edges between nodes 1 and nodes 2, 3, 5 so that $a_{12} = a_{13} = a_{15} = a_{21} = a_{31} = a_{51} = 0$:

$$\mathbf{A} = \begin{bmatrix} 0 & \color{red}{0} & \color{red}{0} & 1 & \color{red}{0} & 1 \\ \color{red}{0} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ \color{red}{0} & a_{32} & 0 & a_{34} & a_{35} & a_{36} \\ 1 & a_{42} & a_{43} & 0 & a_{45} & a_{46} \\ \color{red}{0} & a_{52} & a_{53} & a_{54} & 0 & a_{56} \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & 0 \end{bmatrix}$$

Proceeding like this we obtain the complete adjacency matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Properties of adjacency matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} = [a_{ij}]_{m \times n}$$

- **A** is a *square matrix,* which means that $m = n$.
- All elements a_{ii} on the *(main) diagonal,* aka *diagonal elements,* are zero.
- **A** is *symmetric,* which means that $a_{ij} = a_{ji}$ for all i, j.

We can see here that the adjacency matrix for our particular graph has these properties. But one can show that any adjacency matrix of any undirected loop-free graph has the three properties listed above.

Summary: Graphs and their adjacency matrices

- *Graphs* can be used to visualize connections between certain objects or people.
- These objects or people are represented by the *nodes* or *vertices* of a graph G.
- The *edges* of G represent the connections between the nodes.
- A *loop-free* graph does not have edges from any node to itself.
- A *simple* graph does not have multiple edges between the same pair of nodes.
- In a *undirected* graph the connections are assumed to be two-way and edges can be drawn as line segments.
- Any simple undirected graph G with n vertices can be represented by its *adjacency matrix* $\mathbf{A} = [a_{ij}]_{n \times n}$, where $a_{ij} = 1$ when there is an edge between i and j in G and $a_{ij} = 0$ otherwise.