Lecture 41: Norms and Distances

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MATH3200: Applied Linear Algebra

Review: The standard basis vectors

We have been working with the set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n\}$ of **standard basis vectors** in $\mathbb{R}^n$ a lot.

Question L41.1: Describe in your own words what the vector $\vec{\mathbf{e}}_i$ is when $1 \leq i \leq n$.

The vector $\vec{\mathbf{e}}_i$ has a 1 as its coordinate number i and has the number 0 in all its other coordinates.

Recall that we usually understand $\mathbb{R}^n$ as the set of n-dimensional column vectors, but we also use the symbol for the set of n-dimensional row vectors. For many calculations this distinction is important. However, for most of the material in Chapter 5 it does not matter and we will usually work with row vectors in this chapter for convenience.

Review: Some properties of the standard basis vectors

The set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \dots, \vec{\mathbf{e}}_n\}$ of **standard basis vectors** in $\mathbb{R}^n$ has the following properties:

- The set is *linearly independent.*
- It is a *maximal* linearly independent set in the sense that every larger set of vectors in $\mathbb{R}^n$ will be linearly dependent.
- *Every* vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ can be expressed as a linear combination
  $c_1\vec{\mathbf{e}}_1 + c_2\vec{\mathbf{e}}_2 + \cdots + c_n\vec{\mathbf{e}}_n = \vec{\mathbf{x}}$
  for a *unique* vector of coefficients $[c_1, c_2, \dots, c_n]^T$.
- The coefficients c_i are the **Cartesian coordinates** of $\vec{\mathbf{x}}$.

Only the last property distinguishes $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \dots, \vec{\mathbf{e}}_n\}$ from any other basis of $\mathbb{R}^n$.

Are there other properties that make this set really special?

Three additional properties of standard basis vectors

The set $B = \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n\}$ of **standard basis vectors** in $\mathbb{R}^n$ has the following properties:

- Every vector in this set has *length 1.*
- The vectors in this set are all *orthogonal* that is, at right angles.
- This makes it possible to find the coordinates of any vector $\vec{\mathbf{x}}$ with respect to B as the *projections* of $\vec{\mathbf{x}}$ on the coordinate axes.

In this lecture we will explore general notions of length of a vector called *norms.*

In the next lecture we will explore general notions of *orthogonality.*

We will then see that the nice properties of $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n\}$ listed above are shared by **any orthonormal basis.**

Length of a vector: Norms

Let $\ell(\vec{\mathbf{x}})$ denote the length of a vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$.

This function has the following properties:

(i) $\ell(\vec{\mathbf{x}}) \geq 0 \quad$ and $\quad \ell(\vec{\mathbf{x}}) = 0$ if, and only if, $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(ii) If α is any scalar, then $\ell(\alpha\vec{\mathbf{x}}) = |\alpha|\ell(\vec{\mathbf{x}})$.

(iii) $\ell(\vec{\mathbf{x}} + \vec{\mathbf{y}}) \leq \ell(\vec{\mathbf{x}}) + \ell(\vec{\mathbf{y}})$ for any vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$.

In linear algebra it is customary to call any function $\ell : V \to \mathbb{R}$ that is defined on a vector space V and has the above properties a *norm* on V.

It is also customary to use the notation $\|\vec{\mathbf{x}}\|$ instead of $\ell(\vec{\mathbf{x}})$ for the norm (length) of a vector.

There are many different examples of norms and they have their uses in certain applications. You will learn in Module 81 of several such examples.

However, in the lectures for this chapter, we will always use the standard *Euclidean norm* $\|\vec{\mathbf{x}}\|$ of a vector $\vec{\mathbf{x}}$ that corresponds to our usual notion of length.

It is defined as follows: Let $\vec{\mathbf{x}}$ be a row or column vector with elements $x_1, x_2, \ldots, x_n$. Then

$$\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Question L41.2: What is $\|[1, 2, -2]\|$?

$$\|[1, 2, -2]\| = \sqrt{1^2 + 2^2 + (-2)^2)} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$$

Unit vectors

A *unit vector* with respect to a given norm $\| \cdot \|$ is a vector $\vec{\mathbf{x}}$ such that $\|\vec{\mathbf{x}}\| = 1$. The notation $\| \cdot \|$ means that we treat the norm as a function without specifying an input vector $\vec{\mathbf{x}}$ that will go in the place of the dot.

The standard basis vectors $\vec{\mathbf{e}}_i$ are unit vectors with respect to the Euclidean norm.

For example, consider the vector $\vec{\mathbf{e}}_1 = [1, 0]$ in $\mathbb{R}^2$. Then: $\|[1, 0]\| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$.

Question L41.3: Which of the following vectors are unit vectors with respect to the Euclidean norm: $[1, 1]$, $[3/7, 4/7]$, $[3/5, 4/5]$?

$$\|[1, 1]\| = \sqrt{2}, \quad \|[3/7, 4/7]\| = \sqrt{\tfrac{25}{49}}, \quad \|[3/5, 4/5]\| = \sqrt{\tfrac{25}{25}} = 1.$$

So only $[3/5, 4/5]$ is a unit vector for the Euclidean norm.

Normalization

Every nonzero vector $\vec{\mathbf{x}}$ can be *normalized* to a unit vector by dividing it by its norm: $\frac{\vec{\mathbf{x}}}{\|\vec{\mathbf{x}}\|}$.

This *normalization of* $\vec{\mathbf{x}}$ is a unit vector that points in the same direction.

For example, when we normalize $[5, -12]$ with respect to the Euclidean norm, we have $\|[5, -12]\| = \sqrt{25 + 144} = 13$, and we obtain the unit vector $\left[\frac{5}{13}, \frac{-12}{13}\right]$.

The result of normalization depends on the particular norm we are working with, but in the lectures this will always be the Euclidean norm.

Question L41.4: Find the normalization of the vector $[3, -4]$.

Here $\|[3, -4]\| = \sqrt{3^2 + (-4)^2} = 5$,

so the normalization will be the vector $\frac{1}{5}[3, -4] = [3/5, -4/5]$.

Normalization and MATLAB output

Normalization is the reason why some MATLAB output looks so strange.

For example, the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ has eigenvectors

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ with eigenvalues 0 and 6, respectively.

When you enter

```
>> [vec, val] = eig(A)
```

MATLAB will output a matrix $\begin{bmatrix} -0.7071 & -0.4472 \\ 0.7071 & -0.8944 \end{bmatrix}$

whose columns that are normalizations (with respect to the Euclidean norm) of the nicer-looking eigenvectors listed above.

Given a norm $\|\cdot\|$ on a vector space V, we can define a *distance* $d(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ between any two vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ in V by:

$$d(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\|$$

Again, the distance between two vectors depends on the particular norm we are working with, but in the lectures this will always be the Euclidean norm.

Question L41.5: Let $\vec{\mathbf{x}} = [3, 4, 5]$ and $\vec{\mathbf{y}} = [2, 2, 7]$. Find $d(\vec{\mathbf{x}}, \vec{\mathbf{y}})$.

Here $\vec{\mathbf{x}} - \vec{\mathbf{y}} = [1, 2, -2]$, and it follows that

$$d([3, 4, 5], [2, 2, 7]) = \|[1, 2, -2]\| = \sqrt{1^2 + 2^2 + (-2)^2} = 3.$$

Take-home message

The *norm* of a vector $\vec{\mathbf{x}}$ is a real number $\|\vec{\mathbf{x}}\|$ that can be thought of as its length. It has the following properties:

(i) $\|\vec{\mathbf{x}}\| \geq 0$ and $\|\vec{\mathbf{x}}\| = 0$ if, and only if, $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(ii) If α is any scalar, then $\|\alpha\vec{\mathbf{x}}\| = |\alpha| \, \|\vec{\mathbf{x}}\|$.

(iii) $\|\vec{\mathbf{x}} + \vec{\mathbf{y}}\| \leq \|\vec{\mathbf{x}}\| + \|\vec{\mathbf{y}}\|$ for any vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$.

There are many different norms on a given vector space.
In the lectures we will always use the familiar *Euclidean norm*

$$\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

A *unit vector* wrt (with respect to) a given norm is a vector $\vec{\mathbf{x}}$ such that $\|\vec{\mathbf{x}}\| = 1$.

The *normalization* of $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ wrt a given norm is the vector $\frac{\vec{\mathbf{x}}}{\|\vec{\mathbf{x}}\|}$.

The *distance* $d(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ between vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ is defined as $d(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\|$.