Lecture 42: Inner Products and Orthogonality

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MATH3200: Applied Linear Algebra

The (standard) inner product of two vectors

Let $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ be two vectors of the same order having real components.

Then the *(standard) inner product* $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle$ aka *dot product* is calculated by multiplying the corresponding elements of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ and adding the resulting terms.

For row vectors $\vec{\mathbf{x}} = [x_1, x_2, \dots, x_n], \vec{\mathbf{y}} = [y_1, y_2, \dots, y_n]$ we get:

$\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$.

Example: Let $\vec{\mathbf{x}} = [1, 2, 3], \vec{\mathbf{y}} = [-1, 0, 4]$.

Then $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = (1)(-1) + (2)(0) + (3)(4) = 11$.

Question L42.1: Let $\vec{\mathbf{x}} = [1, 2, -3, 4], \vec{\mathbf{y}} = [8, 2, 1, -1]$.

Find $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle$.

$\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = (1)(8) + (2)(2) + (-3)(1) + (4)(-1) = 5$.

Two remarks on the terminology

We have seen an "inner product" already in Lecture 5.

For two row vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ of the same length we could form the matrix product $\vec{\mathbf{x}}\vec{\mathbf{y}}^T$, which would evaluate to:

$\vec{\mathbf{x}}\vec{\mathbf{y}}^T = [\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle]$.

So the two notions of inner product are practically the same, except that formally $\vec{\mathbf{x}}\vec{\mathbf{y}}^T$ is a 1×1 matrix, while $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle$ is a real number.

The number $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle$ is called the *standard* inner product because there are other ways of calculating *alternative* inner products that behave in similar ways and are useful in some applications.

However, we will not cover these generalizations in this course and restrict our attention to the standard inner product.

Application 1: Matrix products

Consider $\mathbf{A} = [a_{ij}]_{m\times k}$ and $\mathbf{B} = [b_{ij}]_{k\times n}$.

For $i = 1, \dots, m$, let $\vec{\mathbf{a}}_{i*}$ denote the i-th row of $\mathbf{A}$, and

For $j = 1, \dots, n$, let $\vec{\mathbf{b}}_{*j}$ denote the j-th column of $\mathbf{B}$.

Then the matrix product $\mathbf{AB}$ can be expressed as follows:

$$\mathbf{AB} = \begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{k1} & \dots & b_{kn} \end{bmatrix}$$

$$= \begin{bmatrix} \langle \vec{\mathbf{a}}_{1*} \vec{\mathbf{b}}_{*1}^T \rangle & \dots & \langle \vec{\mathbf{a}}_{1*} \vec{\mathbf{b}}_{*n}^T \rangle \\ \vdots & & \vdots \\ \langle \vec{\mathbf{a}}_{m*} \vec{\mathbf{b}}_{*1}^T \rangle & \dots & \langle \vec{\mathbf{a}}_{m*} \vec{\mathbf{b}}_{*n}^T \rangle \end{bmatrix}$$

Application 2: The Euclidean norm and the inner product

Let $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]$ or $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]^T$.

Recall that the Euclidean norm of $\vec{\mathbf{x}}$ is defined as:

$\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Question L42.2: How is $\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle$ related to $\|\vec{\mathbf{x}}\|$?

$\|\vec{\mathbf{x}}\| = \sqrt{\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle}$, or, equivalently, $\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle = \|\vec{\mathbf{x}}\|^2$.

Orthogonality

Consider $\vec{\mathbf{e}}_1 = [1, 0, 0]$, $\vec{\mathbf{e}}_3 = [0, 0, 1]$, $\vec{\mathbf{x}} = [1, -1, 0]$, $\vec{\mathbf{y}} = [1, 1, -1]$.

Let's explore which pairs of the above vectors are perpendicular (aka *orthogonal*) to each other.

Question L42.3: Are $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_3$ orthogonal?

Yes. $\vec{\mathbf{e}}_1$ is on the x-axis, while $\vec{\mathbf{e}}_3$ is on the z-axis.

Question L42.4: Are $\vec{\mathbf{e}}_3$ and $\vec{\mathbf{x}}$ orthogonal?

Yes. $\vec{\mathbf{e}}_3$ is on the z-axis, while $\vec{\mathbf{x}}$ is in the x-y-plane.

Question L42.5: Are $\vec{\mathbf{x}}$ and $\vec{\mathbf{e}}_1$ orthogonal?

No. Both are in the x-y-plane, $\vec{\mathbf{x}}$ is on a diagonal.
Similarly, $\vec{\mathbf{y}}$ and $\vec{\mathbf{e}}_1$ are not orthogonal, and neither are $\vec{\mathbf{y}}$ and $\vec{\mathbf{e}}_3$.

Question L42.6: Are $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ orthogonal?

This is geometrically less obvious.

Application 3: Orthogonality and the inner product

Consider $\vec{\mathbf{e}}_1 = [1, 0, 0]$, $\vec{\mathbf{e}}_3 = [0, 0, 1]$, $\vec{\mathbf{x}} = [1, -1, 0]$, $\vec{\mathbf{y}} = [1, 1, -1]$.

Let us calculate the inner products of pairs of these vectors.

$\langle \vec{\mathbf{e}}_1, \vec{\mathbf{e}}_3 \rangle = (1)(0) + 0^2 + (0)(1) = 0.$

$\langle \vec{\mathbf{x}}, \vec{\mathbf{e}}_3 \rangle = (1)(0) + (-1)(0) + (0)(1) = 0.$

$\langle \vec{\mathbf{e}}_1, \vec{\mathbf{x}} \rangle = (1)(1) + 0(-1) + 0^2 = 1.$

$\langle \vec{\mathbf{e}}_1, \vec{\mathbf{y}} \rangle = (1)(1) + 0(1) + 0(-1) = 1.$

$\langle \vec{\mathbf{y}}, \vec{\mathbf{e}}_3 \rangle = (1)(0) + (1)(0) + (1)(-1) = -1.$

We notice a pattern:

Two vectors $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ are perpendicular aka orthogonal to each other if, and only if, $\langle \vec{\mathbf{u}}, \vec{\mathbf{w}} \rangle = 0$.

Now we calculate: $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = (1)(1) + (-1)(1) + (0)(-1) = 0.$

It follows that $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are orthogonal to each other.

Application 4: Angles between vectors. The Law of Cosines

Theorem

Let $\vec{\mathbf{x}}, \vec{\mathbf{y}} \neq \vec{\mathbf{0}}$ be two nonzero vectors in $\mathbb{R}^n$.
If Θ in $[0, \pi]$ is the angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, then

$$\cos \Theta = \frac{\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle}{\|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\|}.$$

- Note that $\Theta = \frac{\pi}{2}$ if, and only if, $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = 0 = \cos \frac{\pi}{2}$.
- $\Theta = 0$ if, and only if, $\vec{\mathbf{x}} = \alpha \vec{\mathbf{y}}$ for some real number $\alpha > 0$.

  The Law of Cosines implies that this will be the case if, and only if, $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = \|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\| \cos 0 = \|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\|$.
- For $\vec{\mathbf{x}} = [1, -1, 0]$ and $\vec{\mathbf{y}} = \vec{\mathbf{e}}_1$ we get

  $\cos \Theta = \frac{\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle}{\|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\|} = \frac{1}{\sqrt{2}(1)} = \frac{\sqrt{2}}{2} = \cos \frac{\pi}{4}$.

Summary

The *(standard) inner product* $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle$ aka *dot product* of vectors $\vec{\mathbf{x}} = [x_1, x_2, \dots, x_n], \vec{\mathbf{y}} = [y_1, y_2, \dots, y_n]$ is defined as

$\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$.

$\sqrt{\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle} = \|\vec{\mathbf{x}}\|$ is the Euclidean norm of $\vec{\mathbf{x}}$. Thus $\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle = \|\vec{\mathbf{x}}\|^2$.

Two vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ are *orthogonal* if, and only if, $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = 0$.

The angle Θ in $[0, \pi]$ between two nonzero vectors in $\mathbb{R}^n$ can be computed from the Law of Cosines

$$\cos \Theta = \frac{\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle}{\|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\|}.$$