Lecture 43: Orthogonal Projections, Orthogonal Complements, and Orthonormal Bases

Winfried Just
Department of Mathematics, Ohio University

MATH3200: Applied Linear Algebra

Projections and orthogonal complements

Theorem

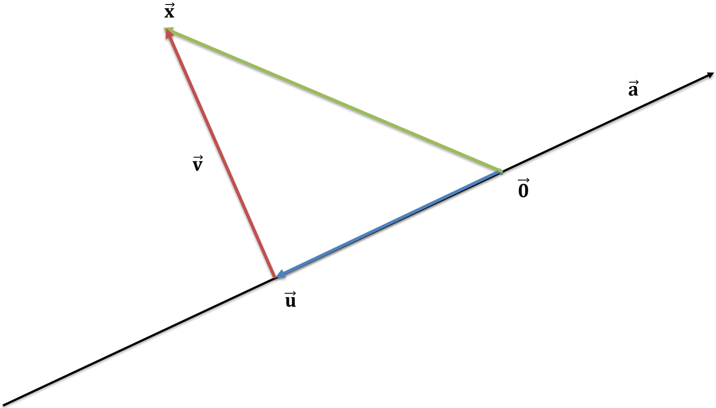
Let $\vec{\mathbf{a}}, \vec{\mathbf{x}}$ *be two vectors in* $\mathbb{R}^n$.

Then there exists exactly one pair $(\vec{\mathbf{u}}, \vec{\mathbf{v}})$ *of vectors in* $\mathbb{R}^n$ *such that*

1. $\vec{\mathbf{x}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$.
2. $\vec{\mathbf{u}} = c\vec{\mathbf{a}}$ *for some scalar* c, *that is,* $\vec{\mathbf{u}}, \vec{\mathbf{a}}$ *are on the same line.*
3. $\langle \vec{\mathbf{a}}, \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = 0$, *that is,* $\vec{\mathbf{v}}$ *is orthogonal to* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{u}}$.

The vector $\vec{\mathbf{u}}$ is called the *projection of* $\vec{\mathbf{x}}$ *onto* $\vec{\mathbf{a}}$, while
$\vec{\mathbf{v}}$ is called the *orthogonal complement of* $\vec{\mathbf{x}}$ *with respect to* $\vec{\mathbf{a}}$.

Projections and orthogonal complements



How to compute projections and orthogonal complements?

Theorem

Let $\vec{\mathbf{a}}, \vec{\mathbf{x}}$ be two vectors in $\mathbb{R}^n$.

Then there exists exactly one pair $(\vec{\mathbf{u}}, \vec{\mathbf{v}})$ of vectors in $\mathbb{R}^n$ such that

1. $\vec{\mathbf{x}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$.
2. $\vec{\mathbf{u}} = c\vec{\mathbf{a}}$ *for some scalar* c, *that is,* $\vec{\mathbf{u}}, \vec{\mathbf{a}}$ *are on the same line.*
3. $\langle \vec{\mathbf{a}}, \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = 0$, *that is,* $\vec{\mathbf{v}}$ *is orthogonal to* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{u}}$.

When $\vec{\mathbf{a}} = \vec{\mathbf{0}}$, *then* $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ *and* $\vec{\mathbf{v}} = \vec{\mathbf{x}}$. *When* $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$, *then*

$$\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}} \qquad \text{and} \qquad \vec{\mathbf{v}} = \vec{\mathbf{x}} - \vec{\mathbf{u}} = \vec{\mathbf{x}} - \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}}.$$

The vector $\vec{\mathbf{u}}$ is called the *projection of* $\vec{\mathbf{x}}$ *onto* $\vec{\mathbf{a}}$, while
$\vec{\mathbf{v}}$ is called the *orthogonal complement of* $\vec{\mathbf{x}}$ *with respect to* $\vec{\mathbf{a}}$.

Example 1 of a projection and an orthogonal complement

Example 1: Consider $\vec{\mathbf{a}} = [1, 2]$ and $\vec{\mathbf{x}} = [-2, 3]$.

The projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{a}}$ is given by

$$\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}} = \frac{\langle [1,2], [-2,3] \rangle}{\langle [1,2], [1,2] \rangle} [1, 2].$$

Question L43.1: What is $\langle [1, 2], [-2, 3] \rangle$?

$\langle [1, 2], [-2, 3] \rangle = 1(-2) + (2)(3) = 4.$

Similarly, $\langle [1, 2], [1, 2] \rangle = (1)(1) + (2)(2) = 5.$

It follows that $\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}} = \frac{4}{5} [1, 2] = [0.8, 1.6].$

The orthogonal complement $\vec{\mathbf{v}}$ of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{a}}$ is given by

$$\vec{\mathbf{v}} = \vec{\mathbf{x}} - \vec{\mathbf{u}} = [-2, 3] - [0.8, 1.6] = [-2.8, 1.4].$$

Example 2 of projections and orthogonal complements

Consider $\vec{\mathbf{x}} = [-2, 3, 4]$.

The projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{a}} = \vec{\mathbf{e}}_1$ is given by

$$\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{e}}_1, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{e}}_1, \vec{\mathbf{e}}_1 \rangle} \vec{\mathbf{e}}_1 = \frac{\langle [1,0,0], [-2,3,4] \rangle}{\langle [1,0,0], [1,0,0] \rangle} \vec{\mathbf{e}}_1 = (-2)\vec{\mathbf{e}}_1 = [-2, 0, 0].$$

The orthogonal complement $\vec{\mathbf{v}}$ of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{e}}_1$ is given by

$$\vec{\mathbf{v}} = \vec{\mathbf{x}} - \vec{\mathbf{u}} = [-2, 3, 4] - [-2, 0, 0] = [0, 3, 4] = 3\vec{\mathbf{e}}_2 + 4\vec{\mathbf{e}}_3.$$

Question L43.2: Find the projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{a}} = \vec{\mathbf{e}}_2$.

$$\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{e}}_2, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_2 \rangle} \vec{\mathbf{e}}_2 = \frac{\langle [0,1,0], [-2,3,4] \rangle}{\langle [0,1,0], [0,1,0] \rangle} \vec{\mathbf{e}}_2 = 3\vec{\mathbf{e}}_2 = [0, 3, 0].$$

Question L43.3: Find the orthogonal complement $\vec{\mathbf{v}}$ of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{e}}_2$.

$$\vec{\mathbf{v}} = \vec{\mathbf{x}} - \vec{\mathbf{u}} = [-2, 3, 4] - [0, 3, 0] = [-2, 0, 4] = (-2)\vec{\mathbf{e}}_1 + 4\vec{\mathbf{e}}_3.$$

Example 2 generalizes

Example 2 illustrates a more general fact. When $\vec{\mathbf{x}}$ is in $\mathbb{R}^n$, then $\vec{\mathbf{x}} = x_1\vec{\mathbf{e}}_1 + \cdots + x_i\vec{\mathbf{e}}_i + \cdots + x_n\vec{\mathbf{e}}_n$ and the vectors $\vec{\mathbf{u}} = x_i\vec{\mathbf{e}}_i$ and $\vec{\mathbf{v}} = x_1\vec{\mathbf{e}}_1 + \cdots + x_{i-1}\vec{\mathbf{e}}_{i-1} + x_{i+1}\vec{\mathbf{e}}_{i+1} + \cdots + x_n\vec{\mathbf{e}}_n$ are orthogonal. Thus we get:

Proposition

Let $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]$ be any vector in $\mathbb{R}^n$. Then the orthogonal projection of $\vec{\mathbf{x}}$ on any standard basis vector $\vec{\mathbf{e}}_i$ is

$\vec{\mathbf{u}} = x_i\vec{\mathbf{e}}_i$.

Moreover, the orthogonal complement is given by

$\vec{\mathbf{v}} = x_1\vec{\mathbf{e}}_1 + \cdots + x_{i-1}\vec{\mathbf{e}}_{i-1} + x_{i+1}\vec{\mathbf{e}}_{i+1} + \cdots + x_n\vec{\mathbf{e}}_n$.

Notice that in this argument we only used the fact that the standard basis vectors $\vec{\mathbf{e}}_1, \ldots, \vec{\mathbf{e}}_n$ are all orthogonal.

Finding the coefficients

Proposition

Let $B = \{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_n\}$ be a basis for a vector space that consists of pairwise orthogonal vectors, that is, a basis with
$\langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_j \rangle = 0$ whenever $i \neq j$.
Let $\vec{\mathbf{c}} = [c_1, \dots, c_n]$ be expressed in alternative coordinates with respect to B.
Then the orthogonal projection of $\vec{\mathbf{c}}$ on any vector $\vec{\mathbf{b}}_i$ is $\vec{\mathbf{u}} = c_i \vec{\mathbf{b}}_i$.
Moreover, the orthogonal complement is given by
$\vec{\mathbf{v}} = c_1 \vec{\mathbf{b}}_1 + \dots + c_{i-1} \vec{\mathbf{b}}_{i-1} + c_{i+1} \vec{\mathbf{b}}_{i+1} + \dots + c_n \vec{\mathbf{b}}_n$.

This observation provides a convenient way of calculating alternative coordinates with respect to a given basis B, as long as all vectors in this basis are orthogonal.

Example 3: Calculating alternative coordinates

Let $B = \{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3\} = \{[1, 1, 1], [-2, 1, 1], [0, 1, -1]\}$.

Here $\langle [1, 1, 1], [-2, 1, 1] \rangle = -2 + 1 + 1 = 0$,
$\langle [1, 1, 1], [0, 1, -1] \rangle = 1 - 1 = 0$, and
$\langle [-2, 1, 1], [0, 1, -1] \rangle = 1 - 1 = 0$.

Thus B is a basis for $\mathbb{R}^3$ that consists of orthogonal vectors.

Let us express the vector $\vec{\mathbf{x}} = [1, 2, 3]$ in alternative coordinates $\vec{\mathbf{c}}$ with respect to B, that is, find coefficients c_1, c_2, c_3 such that $\vec{\mathbf{x}} = c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2 + c_3\vec{\mathbf{b}}_3$.

If we were to use the method of Chapter 3, we would need to write the vectors of the basis B as columns of a matrix $\mathbf{B}$, compute the inverse matrix $\mathbf{B}^{-1}$,

and then obtain the vector $\vec{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{B}^{-1}\vec{\mathbf{x}}$.

Example 3: Calculating alternative coordinates differently

For $B = \{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3\} = \{[1,1,1], [-2,1,1], [0,1,-1]\}$
we have $\langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2 \rangle = \langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_3 \rangle = \langle \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3 \rangle = 0.$

Let us express the vector $\vec{\mathbf{x}} = [1,2,3]$ in alternative coordinates $\vec{\mathbf{c}}$ with repsect to B, that is, let us find coefficients c_1, c_2, c_3 such that
$\vec{\mathbf{x}} = c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2 + c_3\vec{\mathbf{b}}_3.$

Since the vectors in B are all orthogonal, by the observation on slide 8 we can use the simpler method of calculating the alternative coordinates c_1, c_2, c_3 of $\vec{\mathbf{x}}$ with respect to B as the coefficients of the projections of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$, respectively.

In particular,

$c_1 = \frac{\langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_1 \rangle} = \frac{\langle [1,1,1],[1,2,3] \rangle}{\langle [1,1,1],[1,1,1] \rangle} = \frac{6}{3} = 2.$

Example 3: Calculating alternative coordinates, completed

Let $B = \{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3\} = \{[1,1,1], [-2,1,1], [0,1,-1]\}$.

Here $\langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2 \rangle = \langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_3 \rangle = \langle \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3 \rangle = 0$.

Since the vectors in B are all orthogonal, the alternative coordinates c_1, c_2, c_3 of $\vec{\mathbf{x}}$ with respect to B are the coefficients of the projections of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$, respectively.

In particular, $c_1 = \frac{\langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_1 \rangle} = \frac{\langle [1,1,1],[1,2,3] \rangle}{\langle [1,1,1],[1,1,1] \rangle} = \frac{6}{3} = 2$.

Question L43.4: Find c_2.

$c_2 = \frac{\langle \vec{\mathbf{b}}_2, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_2 \rangle} = \frac{\langle [-2,1,1],[1,2,3] \rangle}{\langle [-2,1,1],[-2,1,1] \rangle} = \frac{3}{6} = 0.5$.

Similarly, $c_3 = \frac{\langle \vec{\mathbf{b}}_3, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{b}}_3, \vec{\mathbf{b}}_3 \rangle} = \frac{\langle [0,1,-1],[1,2,3] \rangle}{\langle [0,1,-1],[0,1,-1] \rangle} = \frac{-1}{2} = -0.5$.

Check the result: $2[1,1,1] + 0.5[-2,1,1] - 0.5[0,1,-1] = [1,2,3]$.

Orthonormal bases

In Example 2 we had an added bonus: Since $\langle \vec{\mathbf{e}}_i, \vec{\mathbf{e}}_i \rangle = 1$ for all $\vec{\mathbf{e}}_i$, the direct calculation of the projection was much simpler than in Example 3. The following definition codifies these nice properties of $\{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_n\}$:

Definition

(i) A set $B = \{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_n\}$ of vectors is *orthogonal* if $\langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_j \rangle = 0$ for all $i \neq j$.

(ii) We call B *orthonormal* if it is orthogonal and composed of unit vectors, so that $\|\vec{\mathbf{b}}_i\| = \langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_i \rangle = 1$ for all $i = 1, \dots, n$.

(iii) A set B is an *orthonormal basis* for a linear subspace V of $\mathbb{R}^n$ if it is orthonormal and a basis for V.

In particular $B = \{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

The basis B of Example 3 was an orthogonal basis for $\mathbb{R}^3$, but not an orthonormal basis.

Alternative coordinates with respect to orthonormal bases

Let V be a vector space, and let $B = \{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_k\}$ be an *orthogonal* basis of V. Then any vector $\vec{\mathbf{x}}$ in V can be expressed in alternative coordinates $\vec{\mathbf{c}}$ wrt the this basis B as follows:

$$\vec{\mathbf{c}} = \left[\frac{\langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_1 \rangle}, \dots, \frac{\langle \vec{\mathbf{b}}_k, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{b}}_k, \vec{\mathbf{b}}_k \rangle} \right]. \qquad (1)$$

In particular, when B is *orthonormal,* then

$\langle \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_1 \rangle, \dots, \langle \vec{\mathbf{b}}_k, \vec{\mathbf{b}}_k \rangle = 1$, and (1) simplifies to

$\vec{\mathbf{c}} = \left[\langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle, \dots, \langle \vec{\mathbf{b}}_k, \vec{\mathbf{x}} \rangle \right]$, so that $\vec{\mathbf{x}} = \langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle \vec{\mathbf{b}}_1 + \cdots + \langle \vec{\mathbf{b}}_k, \vec{\mathbf{x}} \rangle \vec{\mathbf{b}}_k$.

In the above notation we have implicitly assumed that V consists of row vectors. The analogue result holds for spaces V that consist of column vectors.

Summary: Projections and orthogonal complements

Theorem

Let $\vec{\mathbf{a}}, \vec{\mathbf{x}}$ be two vectors in $\mathbb{R}^n$.

Then there exists exactly one pair $(\vec{\mathbf{u}}, \vec{\mathbf{v}})$ of vectors in $\mathbb{R}^n$ such that

1. $\vec{\mathbf{x}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$.
2. $\vec{\mathbf{u}} = c\vec{\mathbf{a}}$ *for some scalar* c, *that is,* $\vec{\mathbf{u}}, \vec{\mathbf{a}}$ *are on the same line.*
3. $\langle \vec{\mathbf{a}}, \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = 0$, *that is,* $\vec{\mathbf{v}}$ *is orthogonal to* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{u}}$.

When $\vec{\mathbf{a}} = \vec{\mathbf{0}}$, *then* $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ *and* $\vec{\mathbf{v}} = \vec{\mathbf{x}}$. *When* $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$, *then*

$$\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}} \qquad \text{and} \qquad \vec{\mathbf{v}} = \vec{\mathbf{x}} - \vec{\mathbf{u}} = \vec{\mathbf{x}} - \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}}.$$

The vector $\vec{\mathbf{u}}$ is called the *projection of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{a}}$*, while
$\vec{\mathbf{v}}$ is called the *orthogonal complement of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{a}}$.*

Summary: Orthonormal sets of vectors and bases

- A set $B = \{\vec{\mathbf{b}}_1, \ldots, \vec{\mathbf{b}}_n\}$ of vectors is *orthogonal* if $\langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_j \rangle = 0$ for all $i \neq j$.
- We call B *orthonormal* if it is orthogonal and composed of unit vectors, so that $\|\vec{\mathbf{b}}_i\| = \langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_i \rangle = 1$ for all $i = 1, \ldots, n$.
- A set B is an *orthonormal basis* for a linear subspace V of $\mathbb{R}^n$ if it is orthonormal and a basis for V.
- $B = \{\vec{\mathbf{e}}_1, \ldots, \vec{\mathbf{e}}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.
- When B is an orthonormal basis for a vector space V, then every vector $\vec{\mathbf{x}}$ can be written in the alternative coordinates $\vec{\mathbf{c}}$ with respect to B that are given by

  $$\vec{\mathbf{c}} = [\langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle, \langle \vec{\mathbf{b}}_2, \vec{\mathbf{x}} \rangle, \ldots, \langle \vec{\mathbf{b}}_n, \vec{\mathbf{x}} \rangle].$$