Lecture 4: Some Matrix Operations: Addition, Subtraction, Transpose, and Multiplication by a Scalar

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MATH3200: Applied Linear Algebra

Definition of matrix addition

Let $\mathbf{A} = [a_{ij}]_{m \times n}$ and $\mathbf{B} = [b_{ij}]_{m' \times n'}$ be two matrices. The sum

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]_{m \times n}$$

is defined if, and only if,

- $\mathbf{A}$ and $\mathbf{B}$ are of the same order, that is, $m = m'$ and $n = n'$,
- and for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the entries a_{ij}, b_{ij} can be added.

The second condition will be satisfied if, for example, the entries a_{ij} and b_{ij} are all real numbers or are all matrices of the same order whose elements are real numbers. In this lecture we will from now on assume for simplicity that all entries of all matrices are numbers.

Matrix addition: Examples

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 3 \\ 0.5 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 6 \\ 4.5 & 1 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 & 0.5 \end{bmatrix} = \ \textit{Undefined!}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Properties of matrix addition

We will use the notation $\mathbf{O}$ for a matrix whose entries are all equal to zero.

Question L4.1: Is this a good notation or are there potential problems with it?

This notation does not specify the order of the matrix; we can and will use it if the dimensions are implied by the context.

If the order is not implied by the context, the $m \times n$ matrix that has all entries equal to zero will be denoted by $\mathbf{O}_{m \times n}$.
We call it the *zero matrix (of order $m \times n$)*.

Matrix addition has similar properties as addition of numbers:

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *(commutativity),*
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ *(associativity),*
- $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

In these laws, we implicitly assume that the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{O}$ can be added.

Subtraction of matrices works similarly

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 3 \\ 0.5 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 3.5 & 9 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -4 \\ -1.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 5 & 0.5 \end{bmatrix} = \textit{Undefined!}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \\ -10 & -11 & -12 \end{bmatrix}$$

Air travel

Suppose we have three cities in the U.S. and Europe that are all connected by direct flights. We are interested in the average duration d_{ij} (in hours) of a flight from i to j. A matrix representation of this information might look like this:

$$\mathbf{D} = \begin{bmatrix} 0 & 9 & 12.5 \\ 8 & 0 & 4.5 \\ 11 & 4 & 0 \end{bmatrix}$$

This matrix is not symmetric.

Question L4.2: What is the likely explanation?

The cities are ordered from East to West. Eastbound air travel in the northern hemisphere usually takes advantage of tail wind; westbound travel needs to overcome head wind.

The transpose of a matrix

The *transpose* $\mathbf{A}^T$ of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]_{m\times n}$ is the $n \times m$ matrix $\mathbf{B} = [b_{ji}]_{n\times m}$ such that $b_{ji} = a_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

In our example,

$$\mathbf{D} = \begin{bmatrix} 0 & 9 & 12.5 \\ 8 & 0 & 4.5 \\ 11 & 4 & 0 \end{bmatrix} = [d_{ij}] \qquad \mathbf{D}^T = \begin{bmatrix} 0 & 8 & 11 \\ 9 & 0 & 4 \\ 12.5 & 4.5 & 0 \end{bmatrix} = [c_{ij}]$$

Note that d_{ij} represents the average time of flying from city i to city j, while c_{ij} represents the average time of flying from city j to city i.

Properties of the transpose

- Every matrix $\mathbf{A}$ has a transpose $\mathbf{A}^T$.
- For every matrix $\mathbf{A}$ we have $(\mathbf{A}^T)^T = \mathbf{A}$.

For example, let $\quad \mathbf{A} = \begin{bmatrix} 0 & 9 & 12 \\ 8 & 7 & 4 \end{bmatrix}$

Then $\quad \mathbf{A}^T = \begin{bmatrix} 0 & 8 \\ 9 & 7 \\ 12 & 4 \end{bmatrix} \qquad$ and $\qquad \left(\mathbf{A}^T\right)^T = \begin{bmatrix} 0 & 9 & 12 \\ 8 & 7 & 4 \end{bmatrix}$

Properties of the transpose, continued

- The transpose of an $m \times 1$ column vector is a $1 \times m$ row vector and *vice versa.*

For example, let $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$ and $\vec{\mathbf{y}} = \begin{bmatrix} 0 & 9 & 12 \end{bmatrix}$

Then $\vec{\mathbf{x}}^T = \begin{bmatrix} 0 & 8 \end{bmatrix}$ and $\vec{\mathbf{y}}^T = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$

Properties of the transpose, continued

- A square matrix $\mathbf{A}$ is symmetric if, and only if, $\mathbf{A}^T = \mathbf{A}$.

For example, consider $\mathbf{A} = \begin{bmatrix} 0 & 8 \\ 8 & 9 \end{bmatrix}$ with $\mathbf{A}^T = \begin{bmatrix} 0 & 8 \\ 8 & 9 \end{bmatrix}$

Here $\mathbf{A}$ is symmetric and $\mathbf{A} = \mathbf{A}^T$.

In contrast, consider $\mathbf{A} = \begin{bmatrix} 0 & 7 \\ 8 & 9 \end{bmatrix}$ with $\mathbf{A}^T = \begin{bmatrix} 0 & 8 \\ 7 & 9 \end{bmatrix}$

Here $\mathbf{A}$ is not symmetric and $\mathbf{A} \neq \mathbf{A}^T$.

One more properties of the transpose

- For every square matrix $\mathbf{A}$ the sum $\mathbf{A} + \mathbf{A}^T$ is symmetric.

In our example about air travel,

$$\mathbf{D} + \mathbf{D}^T = \begin{bmatrix} 0 & 9 & 12.5 \\ 8 & 0 & 4.5 \\ 11 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 8 & 11 \\ 9 & 0 & 4 \\ 12.5 & 4.5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 17 & 23.5 \\ 17 & 0 & 8.5 \\ 23.5 & 8.5 & 0 \end{bmatrix}$$

Question L4.3: What does $\mathbf{D} + \mathbf{D}^T$ represent in this example?

The total flying times for round trips.

Multiplication of a matrix by a scalar

In linear algebra, the word "scalar" simply means "number."

For any matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ and scalar λ we can define

$$\lambda \mathbf{A} = [\lambda a_{ij}]_{m \times n} = [a_{ij} \lambda]_{m \times n} = \mathbf{A} \lambda.$$

For example, when $\lambda = 3$ and $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

we get:

$$\lambda \mathbf{A} = 3\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Properties of multiplication of a matrix by a scalar

- For any matrix $\mathbf{A}$ and scalar λ we have $\lambda\mathbf{A}^T = (\lambda\mathbf{A})^T$.

For example, let $\lambda = 3$ and $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Then $\lambda\mathbf{A}^T = 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = 3\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 6 & 12 \end{bmatrix}$

Similarly, $(\lambda\mathbf{A})^T = \left(3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}^T = \begin{bmatrix} 3 & 9 \\ 6 & 12 \end{bmatrix}$

Both calculations give the same result.

Properties of multiplication by a scalar, continued

- For any matrices **A**, **B** of the same order and scalar λ we have $\lambda\mathbf{A} + \lambda\mathbf{B} = \lambda(\mathbf{A} + \mathbf{B})$ *(distributivity).*

For example, let $\lambda = 3$ and $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$

Then

$$\lambda\mathbf{A} + \lambda\mathbf{B} = 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 6 & 15 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 15 & 27 \end{bmatrix}$$

Similarly,

$$\lambda(\mathbf{A} + \mathbf{B}) = 3\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}\right) = 3\begin{bmatrix} 2 & 2 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 15 & 27 \end{bmatrix}$$

Both calculations give the same result.

Properties of multiplication by a scalar, continued

- For any matrices $\mathbf{A}$, $\mathbf{B}$ of the same order we have $\mathbf{A} + (-1)\mathbf{B} = \mathbf{A} - \mathbf{B}$,

For example, let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$

Then

$$\mathbf{A} + (-1)\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

Similarly, $\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

Both calculations give the same result.

One more properties of multiplication by a scalar

- For any matrix $\mathbf{A}$ and scalars λ, κ we have $\kappa(\lambda\mathbf{A}) = (\kappa\lambda)\mathbf{A}$.

For example, let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\lambda = 2$, $\kappa = 3$.

Then $\kappa(\lambda\mathbf{A}) = 3\left(2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 3\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 18 & 24 \end{bmatrix}$

Similarly, $(\kappa\lambda)\mathbf{A} = 6\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 18 & 24 \end{bmatrix}$

Both calculations give the same result.

Summary: Addition and subtraction of matrices

- Let $\mathbf{A} = [a_{ij}]_{m\times n}$ and $\mathbf{B} = [b_{ij}]_{m\times n}$ be two matrices of the same order. We define:
  - $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]_{m\times n}$
  - $\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}]_{m\times n}$
- When $\mathbf{A}$ and $\mathbf{B}$ have *different orders,* then $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$ are *undefined*.
- The $m \times n$ matrix that has all entries equal to zero will be denoted by $\mathbf{O}_{m\times n}$ or simply $\mathbf{O}$ if the dimensions are implied by the context and called the *zero matrix (of order $m \times n$)*.
- Properties of matrix addition:
  - $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *(commutativity),*
  - $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ *(associativity),*
  - $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

Summary: The transpose of a matrix

- The *transpose* $\mathbf{A}^T$ of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is the $n \times m$ matrix $\mathbf{B} = [b_{ji}]_{n \times m}$ such that $b_{ji} = a_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.
- Every matrix $\mathbf{A}$ has a transpose $\mathbf{A}^T$.
- For every matrix $\mathbf{A}$ we have $(\mathbf{A}^T)^T = \mathbf{A}$.
- The transpose of an $m \times 1$ column vector is a $1 \times m$ row vector.
- The transpose of a $1 \times n$ row vector is an $n \times 1$ column vector.
- A square matrix $\mathbf{A}$ is symmetric if, and only if, $\mathbf{A}^T = \mathbf{A}$.
- For every square matrix $\mathbf{A}$ the sum $\mathbf{A} + \mathbf{A}^T$ is symmetric.

Summary: Multiplication of a matrix by a scalar

- In linear algebra, the word "scalar" simply means "number".
- For any matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ and scalar λ we define $\lambda \mathbf{A} = [\lambda a_{ij}]_{m \times n} = [a_{ij} \lambda]_{m \times n} = \mathbf{A} \lambda$.
- For any matrices $\mathbf{A}$, $\mathbf{B}$ of the same order and scalars λ, κ we have:
  - $\lambda \mathbf{A}^T = (\lambda \mathbf{A})^T$.
  - $\lambda \mathbf{A} + \lambda \mathbf{B} = \lambda(\mathbf{A} + \mathbf{B})$ *(distributivity)*.
  - $\mathbf{A} + (-1)\mathbf{B} = \mathbf{A} - \mathbf{B}$.
  - $\kappa(\lambda \mathbf{A}) = (\kappa \lambda)\mathbf{A}$.