Lecture 5: Products of Matrices

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MATH3200: Applied Linear Algebra

Matrix multiplication has a few surprises up its sleeve

Let $\mathbf{A} = [a_{ij}]_{m \times n}, \mathbf{B} = [b_{ij}]_{m' \times n'}$ be two matrices.

The *sum* $\mathbf{A} + \mathbf{B}$ behaves *exactly as one might expect,*
the **product AB** *doesn't.*

- $\mathbf{A} + \mathbf{B}$ is *defined whenever $m = m'$ and $n = n'$.*

  AB *may not be defined for matrices of the same order, and is sometimes meaningful for matrices of different orders.*

- When $\mathbf{A} + \mathbf{B} = [c_{ij}]$, then *always $c_{ij} = a_{ij} + b_{ij}$.*

  When $\mathbf{AB} = [d_{ij}]$, then *usually $d_{ij} \neq a_{ij} b_{ij}$.*

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, *exactly as for addition of numbers.*

  Unlike in multiplication of numbers, it is possible that
  $\mathbf{AB} \neq \mathbf{BA}$.

When can we multiply two matrices?

Let $\mathbf{A} = [a_{ij}]_{k\times n}, \mathbf{B} = [b_{ij}]_{m\times p}$ be two matrices.

Then the product **AB** is defined if, and only if, $n = m$, that is, *the number of columns of* **A** *is equal to the number of rows of* **B**.

$$\text{Let} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 4 \\ -1 & 7 \\ -3 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 5 & 6 & 7 \\ 4 & 3 & 2 \end{bmatrix}$$

AB *is defined* (3 columns, 3 rows),

AC *is not defined* (3 columns, 2 rows),

AA *is defined* (3 columns, 3 rows),

BA *is not defined* (2 columns, 3 rows).

When can we multiply two matrices?

Let $\mathbf{A} = [a_{ij}]_{k \times n}, \mathbf{B} = [b_{ij}]_{m \times p}$ be two matrices.

Then the product $\mathbf{AB}$ is defined if, and only if, $n = m$, that is, *the number of columns of* $\mathbf{A}$ *is equal to the number of rows of* $\mathbf{B}$.

$$\text{Let} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 4 \\ -1 & 7 \\ -3 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 5 & 6 & 7 \\ 4 & 3 & 2 \end{bmatrix}$$

Question L5.1: Which of the matrix products **BB**, **BC**, **CA**, **CB**, **CC** are defined?

BB *is not defined* (2 columns, 3 rows),

BC *is defined* (2 columns, 2 rows),

CA *is defined* (3 columns, 3 rows),

CB *is defined* (3 columns, 3 rows),

CC *is not defined* (3 columns, 2 rows).

The order of the product

Let $\mathbf{A} = [a_{ij}]_{k \times n}$ and $\mathbf{B} = [b_{ij}]_{n \times p}$ be such that the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

Then the product $\mathbf{AB}$ is defined and *has order $k \times p$.*

$$\text{Let} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 4 \\ -1 & 7 \\ -3 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 5 & 6 & 7 \\ 4 & 3 & 2 \end{bmatrix}$$

$\mathbf{AB}$ has order 3×2, and $\mathbf{AA}$ has order 3×3.

Question L5.2: What are the orders of $\mathbf{BC}$, $\mathbf{CA}$, and $\mathbf{CB}$?

$\mathbf{BC}$ has order 3×3,

$\mathbf{CA}$ has order 2×3,

$\mathbf{CB}$ has order 2×2.

Products of matrices with vectors

Let $\mathbf{A}$ be a matrix of order $m \times n$ and let $\vec{\mathbf{v}}$ be a $1 \times m$ row vector.

Then $\vec{\mathbf{v}}\mathbf{A}$ is a $1 \times n$ row vector:

$$\vec{\mathbf{v}}\mathbf{A} = [v_1, \dots, v_m] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = [w_1, \dots, w_n]$$

Now let $\vec{\mathbf{v}}$ be an $n \times 1$ column vector.

Then $\mathbf{A}\vec{\mathbf{v}}$ is an $m \times 1$ column vector:

$$\mathbf{A}\vec{\mathbf{v}} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

The definition of the product

Let $\mathbf{A} = [a_{ij}]_{k \times n}$ and $\mathbf{B} = [b_{ij}]_{n \times p}$ be such that the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{k \times p}$ such that for all $i = 1, \dots, k$ and $j = 1, \dots, p$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{\ell=1}^{n} a_{i\ell}b_{\ell j}.$$

$$\begin{bmatrix} a_{11} \dots a_{1\ell} \dots a_{1n} \\ \dots \\ a_{i1} \dots a_{i\ell} \dots a_{in} \\ \dots \\ a_{k1} \dots a_{k\ell} \dots a_{kn} \end{bmatrix} \begin{bmatrix} b_{11} \dots b_{1j} \dots b_{1p} \\ \dots \\ b_{\ell 1} \dots b_{\ell j} \dots b_{\ell p} \\ \dots \\ b_{n1} \dots b_{nj} \dots b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} \dots c_{1j} \dots c_{1p} \\ \dots \\ c_{i1} \dots \mathbf{c_{ij}} \dots c_{ip} \\ \dots \\ c_{k1} \dots c_{kj} \dots c_{kp} \end{bmatrix}$$

An example of a matrix product

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1, 2$ and $j = 1, 2, 3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

An example of a product: c_{11}

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1, 2$ and $j = 1, 2, 3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{-2} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 1 - 3 + 0 = -2.$$

An example of a product: c_{12}

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1,2$ and $j = 1,2,3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & \mathbf{6} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 4 + 2 + 0 = 6.$$

What is c_{13}?

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1, 2$ and $j = 1, 2, 3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & \mathbf{6} & ? \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Question L5.3: What is c_{13}?

An example of a product: c_{13}

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1, 2$ and $j = 1, 2, 3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 & \mathbf{1} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$$c_{13} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 2 - 1 + 0 = 1.$$

An example of a product: c_{21}

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1, 2$ and $j = 1, 2, 3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 1 \\ ? & c_{22} & c_{23} \end{bmatrix}$$

Question L5.4: What is c_{21}?

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 0 + 6 + 3 = 9.$$

An example of a product: c_{22}

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1, 2$ and $j = 1, 2, 3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 1 \\ 9 & -\mathbf{7} & c_{23} \end{bmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = 0 - 4 - 3 = -7.$$

An example of a product: c_{23}

Let $\mathbf{A} = [a_{ij}]_{2\times 3}$ and $\mathbf{B} = [b_{ij}]_{3\times 3}$.

Then the product $\mathbf{AB}$ is the matrix $\mathbf{C} = [c_{ij}]_{2\times 3}$ such that for all $i = 1, 2$ and $j = 1, 2, 3$:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{\ell=1}^{3} a_{i\ell}b_{\ell j}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 1 \\ 9 & -7 & \mathbf{8} \end{bmatrix}$$

$$c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = 0 + 2 + 6 = 8.$$

The inner product of two vectors

Assume $\vec{\mathbf{x}}$ is a $1 \times n$ row vector and $\vec{\mathbf{y}}$ is an $m \times 1$ column vector. Then $\vec{\mathbf{x}}\vec{\mathbf{y}}$ exists if, and only if, $n = m$, that is, if these vectors have the same dimension.

If the product $\vec{\mathbf{x}}\vec{\mathbf{y}}$ exists, it has order 1×1.

$$\vec{\mathbf{x}}\vec{\mathbf{y}} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [c],$$

where $c = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{\ell=1}^{n} x_\ell y_\ell$

is called the *inner product* or *dot product* of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.

The inner product: Examples

Let

$$\vec{\mathbf{x}} = \begin{bmatrix} 2 & 4 \end{bmatrix} \qquad \vec{\mathbf{y}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad \vec{\mathbf{z}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \vec{\mathbf{u}} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

Then

$\vec{\mathbf{x}}\vec{\mathbf{y}} = [(2)(-1) + (4)(3)] = [10]$.

Note that $\vec{\mathbf{x}}\vec{\mathbf{u}}$ is undefined.

Question L5.5: What is $\vec{\mathbf{x}}\vec{\mathbf{z}}$?

$\vec{\mathbf{x}}\vec{\mathbf{z}} = [(2)(2) + (4)(-1)] = [0]$.

An application of inner products

Sums of vectors can be expressed as inner products:

$$\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{\ell=1}^{n} x_\ell \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

For example, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1+2+3] = [6]$

and $\begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [5+6] = [11]$

A second look at the definition of the product **AB**

$$\begin{bmatrix} a_{11} \dots a_{1\ell} \dots a_{1n} \\ \dots \\ a_{i1} \dots a_{i\ell} \dots a_{in} \\ \dots \\ a_{k1} \dots a_{k\ell} \dots a_{kn} \end{bmatrix} \begin{bmatrix} b_{11} \dots b_{1j} \dots b_{1p} \\ \dots \\ b_{\ell 1} \dots b_{\ell j} \dots b_{\ell p} \\ \dots \\ b_{n1} \dots b_{nj} \dots b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} \dots c_{1j} \dots c_{1p} \\ \dots \\ c_{i1} \dots \mathbf{c_{ij}} \dots c_{ip} \\ \dots \\ c_{k1} \dots c_{kj} \dots c_{kp} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{\ell=1}^{n} a_{i\ell}b_{\ell j}.$$

Let $\vec{\mathbf{a}}_{i*}$ denote the vector in row i of $\mathbf{A}$, and let $\vec{\mathbf{b}}_{*j}$ denote the vector in column j of $\mathbf{B}$.

In this notation, $[\mathbf{c_{ij}}] = \vec{\mathbf{a}}_{i*}\vec{\mathbf{b}}_{*j}$.

Summary

- The product **AB** of two matrices $\mathbf{A} = [a_{ij}]_{k\times n}, \mathbf{B} = [b_{ij}]_{m\times p}$ is defined if, and only if, $n = m$, that is, the number of columns of **A** is equal to the number of rows of **B**.
- If **AB** is defined, it has order $k \times p$.
- If $\mathbf{AB} = [c_{ij}]_{k\times p}$ is defined, then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{\ell=1}^{n} a_{i\ell}b_{\ell j}$.
- The matrix product $\vec{\mathbf{x}}\vec{\mathbf{y}}$ of a row vector $\vec{\mathbf{x}}$ and a column vector $\vec{\mathbf{y}}$ of the same length is a 1×1 matrix whose single element is called *the inner product* or *dot product* of these vectors.
- The sums of the rows of a matrix **A** are given by the matrix product $\mathbf{A}[1\ 1 \ldots 1]^T$ of **A** with a vector of ones.
- Similarly, the sums of the columns are given by the matrix product $[1\ 1 \ldots 1]\mathbf{A}$ of a vector of ones with **A**.