Lecture 7: Some Special Matrices

Winfried Just
Department of Mathematics, Ohio University

MATH3200: Applied Linear Algebra

Doing nothing with numbers and matrices

Adding zero to a number a does nothing: $0 + a = a + 0 = a$.

Similarly, multiplying a number a by 1 does nothing:

$1 \cdot a = a \cdot 1 = a$.

We have already seen the analogue of 0 for matrices:

$$\text{Let} \quad \mathbf{O}_{m\times n} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = [0]_{m\times n}$$

Then $\mathbf{O}_{m\times n} + \mathbf{A} = \mathbf{A} + \mathbf{O}_{m\times n} = \mathbf{A}$
for every matrix $\mathbf{A}$ of order $m \times n$.

We call $\mathbf{O}_{m\times n}$ the *zero matrix* of order $m \times n$. When the order is implied by the context, we simply write $\mathbf{O}$ instead of $\mathbf{O}_{m\times n}$.

Zero vectors

Special cases of zero matrices are *zero vectors,* that is, zero matrices of the form $\mathbf{O}_{m\times 1}$ or $\mathbf{O}_{1\times n}$.

For example, the following matrices are zero vectors:

$$\mathbf{O}_{3\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{O}_{1\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

When we work with zero vectors it will always be clear from the context whether these vectors are supposed to be row vectors or column vectors and what their lengths should be.

So we will simply use the notation $\vec{\mathbf{0}}$ for them and will somewhat informally write "*the* zero vector $\vec{\mathbf{0}}$."

Is there a matrix version of 1?

Is there a matrix $\mathbf{1}$ such that $\mathbf{1A} = \mathbf{A}$ and $\mathbf{A1} = \mathbf{A}$ whenever the products are defined?

You may have guessed by now that the answer is "yes" and that there should be different versions of "$\mathbf{1}$" for different orders.

Question L7.1: What should the matrix version of "$\mathbf{1}$" be when $\mathbf{A}$ is of order 2×2? Let's try some obvious candidates:

(a) $\mathbf{1} = \begin{bmatrix} 1 & 1 \end{bmatrix}$

(b) $\mathbf{1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(c) $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Which option works?

For these candidates for $\mathbf{1}$, let's compute $\mathbf{1A}$ with $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and see how our candidates perform:

(a) $\mathbf{1A} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \end{bmatrix} \neq \mathbf{A}$ *Doesn't work.*

(b) $\mathbf{1A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix} \neq \mathbf{A}$ *Doesn't work.*

(c) $\mathbf{1A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \mathbf{A}$ *This one works!*

Question L7.2: How can we convince ourselves that our third candidate *always* works so that $\mathbf{1A} = \mathbf{A1} = \mathbf{A}$ for *every* 2×2 matrix $\mathbf{A}$?

We need to represent $\mathbf{A}$ with symbols

To see that our third candidate for $\mathbf{1}$ satisfies $\mathbf{1A} = \mathbf{A1} = \mathbf{A}$ for every 2×2 matrix $\mathbf{A}$, we consider an arbitrary matrix $\mathbf{A}$ of order 2×2. We can represent it as $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now we calculate using symbols:

$$\mathbf{1A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a + 0c & 1b + 0d \\ 0a + 1c & 0b + 1d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{A}$$

$$\mathbf{A1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{A}$$

This proves that for our candidate for $\mathbf{1}$ the equalities $\mathbf{1A} = \mathbf{A1} = \mathbf{A}$ hold for *every* 2×2 matrix $\mathbf{A}$.

The official definition of the matrix version of 1

Definition

Let n be a positive integer. Then the *identity matrix* of order $n \times n$ is defined as follows:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

When the order is implied by the context, we will simply write $\mathbf{I}$ instead of $\mathbf{I}_n$.

The notation $\mathbf{I}_{n \times n}$ would be more consistent with the one we use for $\mathbf{O}_{m \times n}$, but we can simplify the notation here as an identity matrix $\mathbf{I}$ must always be a square matrix.

Properties of $\mathbf{I}$

If $\mathbf{A}$ is any $m \times n$ matrix, then $\mathbf{I}_m\mathbf{A} = \mathbf{A}$.

Here $\mathbf{A}$ does not need to be a square matrix. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Similarly, if $\mathbf{A}$ is any $m \times n$ matrix, then $\mathbf{A}\mathbf{I}_n = \mathbf{A}$.

Again, $\mathbf{A}$ does not need to be a square matrix. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Multiplication by a scalar λ *vs.* matrix multiplication by $\lambda\mathbf{I}$

Multiplication of a matrix by a scalar can be treated as a special case of matrix multiplication.

Let λ be a scalar and let $\lambda\mathbf{I} = \mathbf{I}\lambda = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix}$

Then for any matrix $\mathbf{A}$ is of order $m \times n$:

(a) $\lambda\mathbf{A} = \lambda(\mathbf{IA}) = (\lambda\mathbf{I})\mathbf{A}$

(b) $\mathbf{A}\lambda = (\mathbf{AI})\lambda = \mathbf{A}(\mathbf{I}\lambda)$

Question L7.3: What is the order of $\mathbf{I}$ here?

In (a), $\mathbf{I}$ must be $\mathbf{I}_m$, while in (b), $\mathbf{I}$ must be $\mathbf{I}_n$.

How to subtract a scalar from a matrix?

Question L7.4: How can $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \pi$ be computed?

We cannot subtract a number from a matrix, and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - [\pi]$ is also undefined. But

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \pi \mathbf{I} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix} = \begin{bmatrix} 1-\pi & 2 \\ 3 & 4-\pi \end{bmatrix}$$ works fine.

In general, for any $n \times n$ matrix $\mathbf{A}$ we can compute $\mathbf{A} - \lambda \mathbf{I}$:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}$$

Diagonal matrices

The matrices $\lambda\mathbf{I}$ are examples of *diagonal matrices.*

In general, a square matrix $\mathbf{A}$ is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$, that is, when $\mathbf{A}$ is of the form:

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Here are two examples of diagonal matrices other than λI:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix}$$

Is this matrix diagonal?

A square matrix **A** is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$, that is, when **A** is of the form:

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Question L7.5: Is the following matrix diagonal?

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

No, because $c_{ij} = 1 \neq 0$ for $i = 1 \neq j = 3$ and for $i = 3 \neq j = 1$.

Is this matrix diagonal?

A square matrix **A** is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$, that is, when **A** is of the form:

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Question L7.6: Is the following matrix diagonal?

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2018 \end{bmatrix}$$

Yes, because the definition of a diagonal matrix does *not* specify that the diagonal elements λ_i must be nonzero.

Products of diagonal matrices

For diagonal matrices, products are much easier to compute then for other types of matrices. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} (1)(3) + (0)(0) & (1)(0) + (0)(4) \\ (0)(3) + (4)(0) & (0)(0) + (2)(4) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$$

In general, the product of two $n \times n$ diagonal matrices is again an $n \times n$ diagonal matrix that is given by the following formula:

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \kappa_1 & 0 & \dots & 0 \\ 0 & \kappa_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \kappa_n \end{bmatrix} = \begin{bmatrix} \lambda_1\kappa_1 & 0 & \dots & 0 \\ 0 & \lambda_2\kappa_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n\kappa_n \end{bmatrix}$$

Triangular matrices

A square matrix $\mathbf{U} = [u_{ij}]_{n \times n}$ is *upper-triangular* if $u_{ij} = 0$ whenever $i > j$, that is, when u_{ij} sits below the main diagonal.

For example: $$\mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

A square matrix $\mathbf{L} = [\ell_{ij}]_{n \times n}$ is *lower-triangular* if $\ell_{ij} = 0$ whenever $i < j$, that is, when ℓ_{ij} sits above the main diagonal.

For example: $$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix}$$

A matrix is called *triangular* if it is either upper-triangular or lower-triangular.

Note that diagonal matrices are exactly the matrices that are **simultaneously** upper- and lower-triangular.

Transposes of triangular matrices

The transpose of an upper-triangular matrices is a lower-triangular matrix and the transpose of a lower-triangular matrix is an upper-triangular matrix. For our examples:

$$\mathbf{U}^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

$$\mathbf{L}^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

Products of triangular matrices

- The product of two upper-triangular matrices of the same order is again an upper-triangular matrix.
- The product of two lower-triangular matrices of the same order is again a lower-triangular matrix.
- The product of a lower-triangular matrix with an upper-triangular matrix does not have any special properties. In fact, most square matrices are equal to such products of two triangular matrices.

Summary

- $\mathbf{O}_{m\times n}$ denotes the *zero matrix* of order $m \times n$. All of its elements are equal to 0. Zero matrices that are vectors are called *zero vectors* and denoted by $\vec{\mathbf{0}}$.
- The *identity matrix* $\mathbf{I}_n$ is the $n \times n$ matrix whose diagonal elements are all ones and whose off-diagonal elements are all zeros. When the order is implied by the context, we write $\mathbf{I}$ instead of $\mathbf{I}_n$.
- $\mathbf{IA} = \mathbf{A}$ and $\mathbf{AI} = \mathbf{A}$ whenever these products are defined.
- A square matrix $\mathbf{A} = [a_{ij}]_{n\times n}$ is:
  - *diagonal* if $a_{ij} = 0$ whenever $i \neq j$,
  - *upper-triangular* if $a_{ij} = 0$ whenever $i > j$,
  - *lower-triangular* if $a_{ij} = 0$ whenever $i < j$.
  - *triangular* if it is either upper- or lower-triangular.
- Products of two diagonal matrices of the same order can be computed by multiplying the corresponding diagonal elements.